LECTURE 8 (MAY 17)

Concerning Sudakov's minoration inequality, it is worthwhile to note that one of its applications (the Example in the last lecture) provides a method of computing the covering number $N(K, E)$ of any convex body K by any ellipsoid E.

Today we start **Majorizing Measures**. Reference: M. Talagrand, Majorizing measures: the generic chaining, Annals of Probability 1996, vol.24, no.3, 1049–1103.

Definition 1. *Let (T, d) be a metric space. A process $(X_t)_{t\in T}$ is called* **subgaussian** *if it is centered (i.e. $\mathbb{E}X_t = 0$ for all $t \in T$ and*

$$\forall s, t \in T,\ \forall u > 0 : \mathbb{P}\{|X_s - X_t| > u\} \leq 2 \exp\Big(-\frac{u^2}{d(s,t)^2}\Big).$$

Exercise 2. *Check that every Gaussian process is subgaussian under the metric $d(s,t) = (\mathbb{E}|X_s - X_t|^2)^{1/2}$.*

The main problem in the area is to compute, for every subgaussian process, the order of magnitude of $\sup_{t\in T} X_t$. More precisely, to compute

$$\mathbb{E} \sup_{t\in T} X_t.$$

The lower bound (for a Gaussian process) is given by Sudakov's minoration. An upper bound is the so-called Dudley's entropy bound.

Theorem 3. (Dudley's entropy bound). *Let $(X_t)_{t\in T}$ be a subgaussian process. Then*

$$\mathbb{E} \sup_{t\in T} X_t \leq C \int_0^\infty \sqrt{\log N(T, d, \varepsilon)}\, d\varepsilon.$$

Remarks:
1. The integral is finite since the integrand is zero for $\varepsilon > \mathrm{diam} T$.
2. Compare this result with Sudakov's minoration.

The traditional and important method of proving Dudley's entropy bound is **chaining**. Here the main idea of the chaining argument.

The problem easily reduces to bounding for $u > 0$ the tails

$$\mathbb{P}\{\sup_t X_t - X_{t_0} > u\} \leq ?$$

Fix $r > 1$, say $r = 2$. Fix sets Π_j realizing the entropies $N(T, d, r^{-j})$, i.e. Π_j are more and more dense "nets" in T. Let i be the maximal s.t. $r^{-i} \geq \mathrm{diam} T$. So we can assume $\Pi_i = \{t_0\}$ and regard Π_i as the first and the most trivial "net". Let $\pi_j(t)$ be the nearest to t element of Π_j. We then write

$$X_t - X_{t_0} = \sum_{j>i} [X_{\pi_j(t)} - X_{\pi_{j-1}(t)}].$$

Then we bound the summands individually: since $d(\pi_j(t), \pi_{j-1}(t)) \leq 2r^{-j+1}$, each summand can be estimated by the definition of the subgaussian process.