LECTURE 9 (MAY 24)

bf From Dudley's entropy estimate to majorizing measures. We inspect the proof of Dudley's inequality, the chaining argument. We start as in the traditional chaining argument by writing

$$X_t - X_{t_0} = \sum_{j>i} [X_{\pi_j(t)} - X_{\pi_{j-1}(t)}].$$

But the will allow more flexibility in the individual bounds on the summands.

Let $a_j = a_j(t)$, $t \in T$, be positive numbers. Then

$$\mathbb{P}\{\sup_{t\in T} X_t - X_{t_0} > uS\} \le \sum_{j>i} \mathbb{P}\Big\{\exists t \in T:\ X_{\pi_j(t)} - X_{\pi_{j-1}(t)} > ua_j(\pi_j(t))\Big\},$$

where

$$S = \sup_{t\in T} \sum_{j>i} a_j(\pi_j(t)).$$

So we can continue the estimate for the probability as

$$\le \sum_{j>i} \sum_{v\in\Pi_j} \mathbb{P}\Big\{X_v - X_{\pi_{j-1}(v)} > ua_j(v)\Big\}$$

and now estimate the summands by the definition of the subgaussian process. We will get a sum of positive numbers which, for $u = 1$, should sum to something smaller then 1 and, by an appropriate obvious chiice of a_j, are "pieces" of any given probability measure μ.

Working out this argument, we prove

Proposition 1. *If μ is a probability measure on T, then*

$$\mathbb{E}\sup_{t\in T} \le K \sup_{t\in T} \sum_{j>i} r^{-j+1} \sqrt{\log \frac{2}{\mu(\{\pi_j(t)\})}}.$$

Now comes a useful observation. We can freely switch between the language of "nets" (Π_j) and the language of partitions $(\mathcal{A}_j)$.

Indeed, given our nets (Π_j), we construct partitions $(\mathcal{A}_j)$ by taking

$$A_v = \{t \in T:\ \pi_j(t) = v\} \quad \text{for } v \in \Pi_j.$$

We then get

$$\forall A \in \mathcal{A}_j:\ \mathrm{diam} A \le 2r^{-j}.$$

Conversely, if we have partitions $(\mathcal{A}_j)$ satisfying the diameter assumption above, we can simply set $\pi_j(t) = x_{A_j(t)}$, where x_A is an arbitrary fixed point in $A \in \mathcal{A}_j$. Then (Π_j) obtained by this procedure are indeed, good "nets".

We will prove a stronger version of the Proposition above.

Proposition 2. *Let $(\mathcal{A}_j)$ be any sequence of partitions of T satisfying the diameter assumption above. If μ is a probability measure on T, then*

$$\mathbb{E} \sup_{t \in T} \leq K \sup_{t \in T} \sum_{j > i} r^{-j+1} \sqrt{\log \frac{2}{\mu(A_j(t))}}.$$