FINAL HOMEWORK

This final homework is due July 11, 2002. I will be away from Weizmann from June 10 to June 29, and then back here until July 11.

1. Symmetrization of a random variable does not affect its moments. Consider a random variable X, which takes values in a Banach space. Let $\widetilde{X}$ be its simmetrization (see Lecture 2). Prove for $1 \le p < \infty$ that

$$\mathbb{E}\|X\|^p \text{ is finite if and only if } \mathbb{E}\|\widetilde{X}\|^p \text{ is finite.}$$

Hint: use the two-sided inequality for the tail distribution of $\widetilde{X}$ in Lecture 2.

2. Every Gaussian process is realized on a convex set in $\mathbf{R}^n$. Consider the standard Gaussian vector g in $\mathbf{R}^n$ (i.e. $g = (g_1, \dots, g_n)$, where g_i are independent $N(0,1)$ random variables). Let T be a set in $\mathbf{R}^n$. For each $t \in \mathbf{R}^n$, the random variable $X_t = \langle g, t\rangle$ is Gaussian. So, $X = (X_t)_{t\in T}$ is a Gaussian process.

It turns out that this example essentially covers all possible Gaussian processes. Prove this. Namely, consider any Gaussian process $(Y_u)_{u\in U}$ and assume (for simplicity rather than for deep reasons) that the set U is finite. Prove that there exists a process X as in the example above, and there is an onto mapping $\varphi : U \to T$ such that

$$\mathbb{P}\{|Y_u - Y_v| > a\} = \mathbb{P}\{|X_{\varphi(u)} - X_{\varphi(v)}| > a\} \quad \text{for } a > 0 \tag{1}$$

and

$$\mathbb{E} \sup_{u\in U} Y_u = \mathbb{E} \sup_{t\in T} X_t.$$

Hint. The subspace E in L_2 spanned by the functions Y_u, $u \in U$, is a finite dimensional Hilbert space, so E is isometric to $\mathbf{R}^n$ with the euclidean norm. Define the map $\varphi : U \to E$ by letting $\varphi(u) = Y_u$ and define the set K in E by $K = \varphi(U)$. Then (1) will follow from the equality $\mathbb{E}|Y_u - Y_v|^2 = \mathbb{E}|X_{\varphi(u)} - X_{\varphi(v)}|^2$.

In the next two exercises we compare the two known bounds for the supremum of a gaussian process: Dudley's entropy bound

$$\int_0^\infty \sqrt{\log N(T, d, \varepsilon)}$$

and the Majorizing Measure Theorem bound

$$\gamma_2(T,d) = \inf_{\mu} \sup_{t\in T} \int_0^\infty \sqrt{\log \frac{1}{\mu(B(t,\varepsilon))}}\, d\varepsilon$$

(the infimum is over all probability measures on T).

3. The Majorizing Measure bound implies Dudley's entropy bound. Prove that

$$\gamma_2(T,d) \le K \int_0^\infty \sqrt{\log N(T,d,\varepsilon)}$$

with some absolute constant K.

Possible approach: Let $r = 10$. There exists a probability measure μ on T such that for all $j \ge i$ (i is the least integer so that $r^{-i} > \mathrm{diam} T$) we have

$$\mu(\{t_j\}) \ge c(j-i)^{-2} N(T,d,r^{-j})^{-1} \quad \textit{for all } t_j \in N_j,$$

where N_j is the "r^{-j}-net" of T of cardinality $|N_j| = N(T,d,r^{-j})$. With this probability measure, approximate the integral in $\gamma_2(T,d)$ by a Riemann sum

$$\gamma_2(T,d) \le K \sup_{t\in T} \sum_{j\ge i} r^{-j} \sqrt{\log \frac{1}{\mu(B(t,r^{-j}))}}$$

and note that $B(t,r^{-j})$ must contain at least one point t_j from N_j, so the measure of this ball is not less than the measure of $\{t_j\}$, which we know.

4. Dudley's entropy bound does not imply the Majorizing Measure bound. Prove that there exists a metric space (T,d) for which

$$\gamma_2(T,d) < \infty \quad \text{and} \quad \int_0^\infty \sqrt{\log N(T,d,\varepsilon)} = \infty.$$

Possible example: Let e_k, $k = 1,2,\dots$ be orthogonal norm one vectors in a Hilbert space H (for example, the unit coordinate vectors in l_2). Consider the set

$$T = \Big\{ \frac{e_k}{\sqrt{\log k}},\ k = 2,3,\dots \Big\}$$

in H under the Hilbert space metric d. Then

1) $\gamma_2(T,d) < K$ for some absolute constant K (deduce this from the majorizing measure theorem considering the Gaussian process on T from Exercise 2)

2) $\int_0^\infty \sqrt{\log N(T,d,\varepsilon)} = \infty$ (prove that $\log N(T,d,\varepsilon) \sim 1/\varepsilon^2$).