Final Exam

MAT 202, Winter 2004. Due 03/23/2004

1. Prove the following "two out of three" result. Let X be a normed space and Y be its subspace. If any two of X, Y, X/Y are complete, so is the third.

2. Let $(x_k)_{k=1}^{\infty}$ be a sequence of nonzero vectors in a Banach space X. Prove that (x_k) is a Schauder basis of X if and only if the following two conditions hold:

(i) The sequence $(x_k)_{k=1}^{\infty}$ is complete in X, i.e. the closure of $\mathrm{Lin}(x_k)_{k=1}^{\infty}$ coincides with X;

(ii) There is a constant $C > 0$ such that, for every choice of scalars $(a_k)_{k=1}^{\infty}$ and integers $n < m$, we have

$$\Big\| \sum_{k=1}^{n} a_k x_k \Big\| \le C \Big\| \sum_{k=1}^{m} a_k x_k \Big\|.$$

Hint: to prove the sufficiency, it is enough in view of (i) to show that the set Y of all elements of the form $\sum_{k=1}^{\infty} a_k x_k$ is closed in X. This can be shown using (ii), which states that the natural projections in Y onto the finite dimensional subspaces $\mathrm{Lin}((x_k)_{k=1}^{n})$ are all uniformly bounded.

3. (i) Show that the wavelet basis obtained from the Haar wavetet[1] forms a Schauder basis of $L_p[0,1]$ for all $1 \le p < \infty$. *Hint: You might want to use Problem 2 by showing (by induction) that (ii) holds with $C = 1$.*

(ii) Prove that the monomials $1, t, t^2, t^3, \ldots$ do not form a Schauder basis in $C[0,1]$. Conclude that (ii) in Problem 2 can not in general be replaced by a weaker (and more familiar) assumption "$\sum_{k=1}^{\infty} a_n x_n = 0$ implies that all $a_n = 0$".

[1] Recall that the Haar wavelet $\varphi(t)$ equals 1 for $0 \le t < 1/2$ and -1 for $1/2 \le t \le 1$ and 0 elsewhere. Its dilations and translations form the "wavelet basis" that consists of the functions $\phi_{j,k}(t) = \varphi(2^j t - k)$ with integers $j \ge 0$ and $0 \le k < 2^j$.