Homework 1

MAT 202, Winter 2004. Due 01/30/2004

1. (Complementary subspaces) Let X_1 be a subspace of a linear vector space X. Prove that there exists a subspace $X_2 \subseteq X$ with the following two properties:

$$X_1 \cap X_2 = \{0\}, \quad \mathrm{Lin}(X_1 \cup X_2) = X.$$

2. (Additivity of ranges of linear operators?) Let $T_1, T_2 : X \to Y$ be two linear operators and $A \subseteq X$ be a subset. What are relations between

$$(T_1 + T_2)(A) \quad \text{and} \quad T_1(A) + T_2(A)$$

(Minkowski sum in the right hand side). Is the left side always included in the right side? Or vice versa? Or perhaps they are equal? What if A is a subspace?

3. (Hyperplanes vs. linear functionals) Prove that the kernels of two linear functionals f and g coincide, then the functionals are equal up to a scalar factor: $f(x) = cg(x)$.

4. (Uniqueness of extensions) Hahn-Banach theorem is a statement about the existence of an extension of a linear functional. Is such extension always unique? (Prove or find a counterexample)

5. (Complex Hahn-Banach theorem) In Hahn-Banach theorem, the field of scalars was $\mathbb{R}$, not $\mathbb{C}$. Why? Try to find and prove any generalization of Hahn-Banach theorem for the complex case.

6. (Banach limit) This is an extension of the notion of limit to possibly diverging sequences. Prove that to **every** bounded sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers one can assign a real number called $\mathrm{Lim}\{x_n\}$ with the following classical properties:

- If $\{x_n\}$ has a limit, then $\mathrm{Lim}\{x_n\} = \lim\{x_n\}$. In any case (even for diverging sequences), $\inf\{x_n\} \leq \mathrm{Lim}\{x_n\} \leq \sup\{x_n\}$.
- $\mathrm{Lim}\{x_n + y_n\} = \mathrm{Lim}\{x_n\} + \mathrm{Lim}\{y_n\}$; for a constant c one has $\mathrm{Lim}\{cx_n\} = c\,\mathrm{Lim}\{x_n\}$.

You may use either the existence theorem of invariant mean on a commutative semigroup (think of the semigroup to choose here) or use Hahn-Banach theorem as Rudin suggests in his book (look up "Banach limit").

Does the Banach limit that you have constructed always equal to one of cluster points of the sequence?

7. (Invariant measure on all subsets of $\mathbb{R}$) Construct a measure defined on all subsets of $\mathbb{R}$, which is invariant with respect to translations, and such that the measure of any interval equals its length. You may want first to construct an invariant measure I on $\mathbb{T}$ such that $I(f) = \int_{\mathbb{T}} f(y)\, d\lambda$ for all Lebesgue intergrable f. to prove this, modify the proof of the existence theorem on the invariant mean for Y being the subpace of all bounded Lebesgue integrable functions rather than constants.