Homework 3

MAT 202, Winter 2004. Due 02/27/2004

1. (Closedness of kernels and ranges?) Prove that for every operator $T \in L(X, Y)$ acting between Banach spaces, the range of T is closed, while the kernel of T may be not closed. (For the latter, you may want to consider on $C[0,1]$ the operator of integration $(Tf)(t) = \int_0^t f(x)\, dx$.

2. (Continuity and kernels) Prove that for a linear functional f on a Banach space X the following are equivalent:

(i) f is continuous;

(ii) the kernel of f is closed in X;

(iii) the kernel of f is not dense in X.

3. (Hahn-Banach Theoerm for linear operators) Prove that in the Hahn-Banach theorem on extensions of linear functionals with the same norm, one can replace the field of scalars by the Banach space ℓ^∞ that consists of all bounded sequences on $\mathbb{N}$ with the usual supremum norm. Namely, if Y is a subspace of a Banach space X, then every $T \in L(Y, \ell^\infty)$ admits an extension $\widehat{T} \in L(X, \ell^\infty)$ such that $\|\widehat{T}\| = \|T\|$.

4. (Openness is essential in the Separation Theorem) The openness of one of the sets is essential in the geometric form of Hahn-Banach Theorem (the separation theorem for two disjoint convex subsets in a normed space X). Show this for a two-dimensional space X.

5. (Duality between quotients and subspaces) Show that for a subspace Y of a Banach space X

$$(X/Y)^* = Y^\perp,$$

where this identity is given by some isometry. (In fact, show that this isometry is the adjoint of the quotient map $X \to X/Y$).