Homework 3

1. (Closedness of kernels and ranges?) Prove that for every operator $T \in L(X, Y)$ acting between Banach spaces, the range of $T$ is closed, while the kernel of $T$ may be not closed. (For the latter, you may want to consider on $C[0,1]$ the operator of integration $(Tf)(t) = \int_0^t f(x) \, dx$.

2. (Continuity and kernels) Prove that for a linear functional $f$ on a Banach space $X$ the following are equivalent:
   (i) $f$ is continuous;
   (ii) the kernel of $f$ is closed in $X$;
   (iii) the kernel of $f$ is not dense in $X$.

3. (Hahn-Banach Theorem for linear operators) Prove that in the Hahn-Banach theorem on extensions of linear functionals with the same norm, one can replace the field of scalars by the Banach space $\ell^\infty$ that consists of all bounded sequences on $\mathbb{N}$ with the usual supremum norm. Namely, if $Y$ is a subspace of a Banach space $X$, then every $T \in L(Y, \ell^\infty)$ admits an extension $\hat{T} \in L(X, \ell^\infty)$ such that $\|\hat{T}\| = \|T\|$.

4. (Openness is essential in the Separation Theorem) The openness of one of the sets is essential in the geometric form of Hahn-Banach Theorem (the separation theorem for two disjoint convex subsets in a normed space $X$). Show this for a two-dimensional space $X$.

5. (Duality between quotients and subspaces) Show that for a subspace $Y$ of a Banach space $X$

$$(X/Y)^* = Y^\perp,$$

where this identity is given by some isometry. (In fact, show that this isometry is the adjoint of the quotient map $X \to X/Y$).