Homework 4

MAT 202, Winter 2004. Due 03/12/2004

1. (Isomorphisms are stable under duality) Let $T \in L(X,Y)$, where both X and Y are Banach spaces. Prove that T is an isomorphism if and only if T^* is an isomorphism.

2. (Inverse Mapping theorem implies Open Mapping Theorem) The Inverse Mapping Theorem is a simple consequence of the Open Mapping Theorem. Show the converse – deduce the Open Mapping Theorem from the Inverse Mapping Theorem using the injectivisation of a linear operator.

3. (Equivalent norms) Assume that a linear space X is equipped with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$, and X is complete under both norms. Prove that one sided estimate

$$\|x\|_1 \le \|x\|_2 \quad \text{for all } x \in X$$

implies the two sided estimate

$$c\|x\|_2 \le \|x\|_1 \le \|x\|_2 \quad \text{for all } x \in X$$

valid for some positive constant c. Such norms are called equivalent.

4. (Injective vs. bounded below) (a) If an operator is bounded below, then it is injective. Show that the converse does not hold in general *(hint: you may want to look at Problem 1 in Homework 3)*

(b) Consider the operator of multiplication by a fixed function. More precisely, fix a $g \in C[0,1]$ and define $T_g : C[0,1] \to C[0,1]$ by $T_g(f) = f \cdot g$. Prove that T is a continuous linear operator, and describe g for which T is (i) injective, (ii) bounded below.

5. (Complemented subspaces) (a) Every finite-dimenisonal subspace of a Banach space is complemented;

(b) if such a subspace is one-dimensional, then the projection P onto it can be found of norm one;

(c) The subspace of all continuous even functions is complemented in $C[-1,1]$.