Midterm Exam

MAT 235C, Spring 2004. Due 05/28/2004

1. (a) Estimate the tail probability $\mathbb{P}(|\sum a_i X_i| > t)$ for independent mean zero identically distributed random variables X_i, which are uniformly bounded: $|X_i| \leq M$. Here (a_i) is a fixed finite secuence of real numbers and $t > 0$.

(Hint: apply symmetrization and use Laplace transform method similarly to what we did for symmetric Bernoulli r.v.'s)

(b) From (a), deduce Khinchine inequality for these random variables and for $1 \leq p < \infty$:

$$A_p(M)\Big(\sum a_i^2\Big)^{1/2} \leq \Big(\mathbb{E}\Big|\sum a_i X_i\Big|^p\Big)^{1/p} \leq B_p(M)\Big(\sum a_i^2\Big)^{1/2}.$$

(Hint: there are two nontrivial cases: $p = 1$ and $2 < p \leq \infty$. Consider the latter case first, using part (a). Then look at $p = 1$ using symmetrization and extrapolation from second and fourth moments to the first moment. One can also use Paley-Zygmund inequality.)

2. Prove the second Lévy's inequality: for every sequence of independent random variables $X_1, \ldots, X_n$ and every $t > 0$,

$$\mathbb{P}(\max_{k \leq n} |X_k| > t) \leq 2\mathbb{P}\Big(|\sum_{k \leq n} X_k| > t\Big).$$

3. (a) Prove that doubling the argument on the complex circle is an ergodic and measure preserving transformation. Precisely, look at $\Omega = \{\omega \in \mathbb{C} : |\omega| = 1\}$ with normalized Lebesgue measure and define the transformation $T : \Omega \to \Omega$ by $T\omega = \omega^2$. Prove that T is measure preserving and ergodic (compare the rotations of the circle; also think of what this means when we apply Ergodic theorem).

(Hint: for the ergodicity, look at an equivalent model, $\Omega = (0, 1)$ and $T\omega = 2\omega(\mod 1)$. Understand how T acts on binary expansions $\omega = .d_1 d_2 d_3 ...$ where each $d_i = 0$ or 1. For a set $A \subset \Omega$ of positive measure, find a dyadic interval which is almost entirely occupied by A (similarly to what we did for rotations). Deduce that A has almost full measure.)

(b) Describe all measure preserving transformations on a finite set with uniform measure (say, $\Omega = \{1, \ldots, n\}$ and $\mathbb{P}(\{k\}) = 1/n$ for all k).

(c) In the definition of an invariant set, let us change $\mathbb{P}(A) = 0$ or 1 by $A = \emptyset$ or Ω. Show that the rotations on the circle will no longer be ergodic.