201B MIDTERM EXAM February 18, 2005

SOLUTIONS

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1 (a) $$\|P_U(x)\|^2 = \left\|\sum_{n=1}^{\infty} \langle u_n, x\rangle u_n\right\|^2 = \sum_{n=1}^{\infty} \|\langle u_n, x\rangle u_n\|^2$$ by the orthogonality (Lemma 6.23)

$$= \sum_{n=1}^{\infty} |\langle u_n, x\rangle|^2$$ by the orthonormality ($\|u_n\|=1$)

$$\le \|x\|^2$$ by Bessel's inequality.

Hence $\|P_U(x)\| \le \|x\|$ for all $x \in H$, (*)

thus P is bounded.

P_U is linear because

$$P_U(ax+by) = \sum_{n=1}^{\infty} \langle u_n, ax+by\rangle u_n$$

$$= \sum_{n=1}^{\infty} \left(a\langle u_n, x\rangle u_n + b\langle u_n, y\rangle u_n\right)$$ by the linearity of the inner product

$$= a\sum_{n=1}^{\infty} \langle u_n, x\rangle u_n + b\sum_{n=1}^{\infty}\langle u_n, y\rangle u_n$$

$$= a\,P_U(x) + b\,P_U(y).$$

(b) From (*) in part (a), we know that $\|P_U\| \le 1$.

Also, $\forall n$ $P_U(u_n) = \langle u_n, u_n\rangle u_n$ by the orthogonality

$= u_n$

hence $\|P_U(u_n)\| = \|u_n\|$, thus $\boxed{\|P_U\| = 1}$.

(c) $P_U^2 x = P_U\left(\sum_{n=1}^{\infty} \langle u_n, x\rangle u_n\right)$

$= \sum_{n=1}^{\infty} \langle u_n, x\rangle P_U(u_n)$ by the linearity and the continuity of P_U

$= \sum_{n=1}^{\infty} \langle u_n, x\rangle u_n$ by part (b) $\quad (P_U(u_n) = u_n)$

$= P_U x.$

Hence $P_U^2 = P_U$

(d) When U is an orthonormal basis, i.e. when U is complete.

~~This happens~~

Indeed, $(P_U x = x \ \forall x)$ is nothing else than the identity

$$x = \sum_n \langle u_n, x\rangle u_n \qquad \forall x \in H$$

which is equivalent to the completeness of U by Theorem 6.26.

(2) (a) Let $\mathbb{1}_{[a,b]}$ denote the function that takes value 1 on $[a,b]$ and zero ~~other~~ elsewhere. Then

$$\sum_{n=1}^{\infty} \left| \int_a^b u_n(x)\,dx \right|^2 = \sum_{n=1}^{\infty} \left| \int_0^1 \mathbb{1}_{[a,b]}(x)\, \overline{u_n(x)}\, dx \right|^2$$

$$= \sum_{n=1}^{\infty} |\langle u_n, \mathbb{1}_{[a,b]}\rangle|^2$$

$$= \| \mathbb{1}_{[a,b]} \|^2_{L^2[0,1]} \quad \text{by Parseval's equality}$$

$$= \int_0^1 \mathbb{1}^2_{[a,b]}(x)\,dx = \int_a^b dx = b - a.$$

(b) <u>First solution</u>: Assume $U = (u_n)$ is not complete,

then $U^\perp$ is not empty, hence $\exists\, v \in H$: $\langle u_n, v\rangle = 0$, $\|v\| = 1$.

Thus $U \cup \{v\}$ is an orthonormal set.

By Bessel's inequality,

$$\left(\sum_n |\langle u_n, f\rangle|^2\right) + |\langle v, f\rangle|^2 = \|f\|^2 \qquad \forall\, f \in L^2[0,1]$$

In light of part (a),

The condition (*) in Problem 2 can be stated as

$$\sum_n |\langle u_n, f\rangle|^2 = \|f\|^2 \quad \text{for } f = \mathbb{1}_{[a,b)} \quad (\forall\ 0 < a < b < 1).$$

These two identities imply $|\langle v, f\rangle|^2 = 0$ for $f = \mathbb{1}_{[a,b]}$, hence

$$\langle v, f\rangle = 0 \quad \text{for } f = \mathbb{1}_{[a,b]}, \quad \forall\, 0 < a < b < 1.$$

By the linearity, summing up these for different a, b yields

$$\langle v, f\rangle = 0 \quad \forall \text{ piecewise constant function } f$$

In particular,

$$\langle v, h_n\rangle = 0 \quad \forall n, \text{ where } (h_n) \text{ is the Haar wavelet basis}$$

(because ~~Haar~~ the elements of that basis are piecewise-constant).

Thus $$v = \sum \underbrace{\langle h_n, v\rangle}_{=0} v_n = 0$$

Contradiction to $\|v\| = 1$. Q.E.D.

(b) Second solution   let $\mathcal{U} = (u_n)_{n\in\mathbb{N}}$

In light of part (a), we can write condition $(*)$ as

$$\sum_n |\langle u_n, f\rangle|^2 = \|f\|^2 \qquad \forall\ f = \mathbb{1}_{[a,b]},\ 0<a<b<1.$$

Since the left hand side equals $\|P_{\mathcal{U}}(f)\|^2$, we have

$$\|P_U(f)\|^2 = \|f\|^2.$$

Since $f = \underbrace{P_U(f) + (f - P_U(f))}_{\text{orthogonal}}$, we have $\|f\|^2 = \|P_U(f)\|^2 + \|f - P_U(f)\|^2$,

hence

$$\|f - P_U(f)\|^2 = 0$$

thus

$$(\bigstar) \qquad (id - P_U) f = 0 \qquad \text{for } f = \mathbb{1}_{[a,b]},\ 0<a<b<1$$

where id is the identity operator on $L^2[0,1]$.

By the linearity of the operator $(id - P_U)$, we can sum $(\bigstar)$ with different a, b's so as to get

$$(\bigstar\bigstar) \qquad (id - P_U) f = 0 \quad \text{for all piecewise-constant } f.$$

Since the set of piecewise-constant functions is dense in $L^2[0,1]$, any function $g \in L^2[0,1]$ is the limit of some sequence of piecewise-const. functions f_k in $L^2[0,1]$.
Hence the continuity of the operator $(id - P_U)$ (Problem 1a) ~~implies~~ and $(\bigstar\bigstar)$ implies

$$(id - P_{\mathcal{U}}) g = 0 \qquad \forall\ g \in L^2[0,1].$$

That is, $P_U g = g \quad \forall\ g \in L^2[0,1]$
which means that $\mathcal{U}$ is an orthonormal basis
(by the definition of P_U).

(3) (a) Since $u(\cdot,t) \in L^2(\mathbb{T})$, we can write its Fourier series $\forall t$ as

$$u(x,t) = \sum_{n=-\infty}^{\infty} u_n(t) e^{inx}$$

By the PDE,

$$u_t = u_{xxx}$$

$$\sum_{n=-\infty}^{\infty} u_n'(t) e^{inx} = \sum_{n=-\infty}^{\infty} (in)^3 u_n(t) e^{inx}, \quad \text{thus}$$

$$u_n'(t) = -in^3 u_n(t).$$

Solving the O.D.E. $\dot{z} = -in^3 z$ we have $z = ce^{-in^3 t}$, hence

$$u_n(t) = c_n e^{-in^3 t}. \quad \text{for some constants } c_n$$

Hence $u(x,t) = \sum_{n=-\infty}^{\infty} c_n e^{-in^3 t} e^{inx}$

Using the initial condition, $\left(f(x) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx}, \quad \hat{f}_n = \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-inx} dx \right)$

$$u(x,0) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx},$$

we find that $c_n = \hat{f}_n$, thus the solution is

$$\boxed{u(x,t) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{in(x-n^2 t)}}$$

(The part on g^t was not graded).

(b) $u(\cdot,t) \in C^1(\mathbb{T})$ if $u(\cdot,t) \in H^{3/2+\varepsilon}(\mathbb{T})$ for some $\varepsilon > 0$

which happens if

$$\sum_{n=-\infty}^{\infty} n^{3+\varepsilon} |\hat{f}_n|^2 < \infty \quad \text{for some } \varepsilon > 0.$$