1. Let $U = \{u_n : n \in \mathbb{N}\}$ be an orthonormal set in a Hilbert space $H$. Let us define a map $P_U : H \to H$ as

$$P_U(x) = \sum_{n=1}^{\infty} \langle u_n, x \rangle u_n.$$ 

(a) Prove that $P_U$ is a bounded linear operator in $H$. 
(b) Compute the norm of $P_U$. 
(c) Prove that $P_U^2 = P_U$. 
(d) When does $P_U$ equal the identity operator on $H$? Explain.

2. (a) Let $U = \{u_n : n \in \mathbb{N}\}$ be an orthonormal basis of $L^2([0,1])$. Prove that

$$\sum_{n=1}^{\infty} \left| \int_a^b u_n(x) \, dx \right|^2 = b - a \quad \text{for all } 0 < a < b < 1.$$ 

(b) Let $U = \{u_n : n \in \mathbb{N}\}$ be an orthonormal set in $L^2([0,1])$. Assume that condition (*) holds. Prove that $U$ is an orthonormal basis of $L^2([0,1])$. 
(Hint: deduce from Parseval’s identity from (*), first for all piecewise-constant functions, then for all functions using the result in Problem 1a.)

3. Consider the following initial value problem for $u(x,t)$, where $u(\cdot,t) \in L^2(\mathbb{T})$:

$$u_t = u_{xxx},$$
$$u(x,0) = f(x) \in L^2(\mathbb{T}).$$

(This PDE is called linearized KdV (Korteweg-de Vries) equation.) 
(a) Use Fourier series to solve this problem for $t > 0$. 
(b) Find a nontrivial sufficient condition for the solution $u(\cdot,t)$ to be smooth (that is, continuously differentiable) for all $t > 0$. 

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Closed book, closed notes. You may use any standard theorem, provided you state it carefully.