201B MIDTERM EXAM February 18, 2005

Closed book, closed notes. You may use any standard theorem, provided you state it carefully.

1. Let $U = \{u_n : n \in \mathbb{N}\}$ be an orthonormal set in a Hilbert space H. Let us define a map $P_U : H \to H$ as

$$P_U(x) = \sum_{n=1}^{\infty} \langle u_n, x \rangle \, u_n.$$

(a) Prove that P_U is a bounded linear operator in H.
(b) Compute the norm of P_U.
(c) Prove that $P_U^2 = P_U$.
(d) When does P_U equal the identity operator on H? Explain.

2. (a) Let $U = \{u_n : n \in \mathbb{N}\}$ be an orthonormal **basis** of $L^2([0,1])$. Prove that

$$\sum_{n=1}^{\infty} \Big| \int_a^b u_n(x)\, dx \Big|^2 = b - a \quad \text{for all } 0 < a < b < 1. \tag{*}$$

(b) Let $U = \{u_n : n \in \mathbb{N}\}$ be an orthonormal **set** in $L^2([0,1])$. Assume that condition (*) holds. Prove that U is an orthonormal **basis** of $L^2([0,1])$.
(*Hint: deduce from Parseval's identity from (*), first for all piecewise-constant functions, then for all functions using the result in Problem 1a.*)

3. Consider the following initial value problem for $u(x,t)$, where $u(\cdot,t) \in L^2(\mathbb{T})$:

$$u_t = u_{xxx},$$
$$u(x,0) = f(x) \in L^2(\mathbb{T}).$$

(This PDE is called linearized KdV (Korteweg-de Vries) equation.)
(a) Use Fourier series to solve this problem for $t > 0$.
(b) Find a nontrivial sufficient condition for the solution $u(\cdot,t)$ to be smooth (that is, continuously differentiable) for all $t > 0$.