201C FINAL EXAM SOLUTIONS

Spring 2005

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(1) (b) We use the identity

$$\widehat{\chi_{[-R,R]}}(k) = \sqrt{2\pi}\ \frac{\sin(Rk)}{\pi k} \qquad (*)$$

(straightforward calculation)

For $R=1$, $\qquad \widehat{\chi}_{[-1,1]}(x) = \sqrt{\frac{2}{\pi}}\ \frac{\sin x}{x}$

Plancherel's Theorem says that

$$\underbrace{\int_{\mathbb{R}} |\widehat{\chi}_{[-1,1]}(x)|^2\, dx}_{\| \atop \frac{2}{\pi} \int_{\mathbb{R}} \left(\frac{\sin x}{x}\right)^2 dx} = \underbrace{\int_{\mathbb{R}} |\chi_{[-1,1]}(x)|^2\, dx}_{\| \atop 2}$$

Then $\boxed{\int_{\mathbb{R}} \left(\frac{\sin x}{x}\right)^2 dx = \pi.}$

(c) We have to prove that, for every $\varphi \in S(\mathbb{R})$,

$$\langle f_n, \varphi \rangle \to \varphi(0) \quad \text{as } n \to \infty,$$

where $f_n = \frac{\sin(nx)}{\pi x}$.

By $(*)$, $\quad f_n = \widehat{g_n} \quad$ for $\quad g_n = \frac{1}{\sqrt{2\pi}} \chi_{[-n,n]}$.

Hence

$$\langle f_n, \varphi \rangle = \langle \widehat{g}_n, \varphi \rangle = \langle g_n, \widehat{\varphi} \rangle$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi_{[-n,n]}(k)\, \widehat{\varphi}(k)\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} \widehat{\varphi}(k)\, dk \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{\varphi}(k)\, dk$$

$= \varphi(0)$ by the inversion formula.

Q.E.D.

(2) λ is an eigenvalue of A $(Au=u')$ if and only if the equation $u'-\lambda u=0$ has a solution $u\in D(A)$.

The general solution of this equation is

$$u(x)=Ce^{\lambda x}. \qquad (*)$$

If $D(A)=H^1((0,1))$, then $u\in D(A)$ $\forall\lambda$, hence

$$\boxed{\sigma_{point}(A_1)=\mathbb{C}}$$

If $D(A)=\{u\in H^1: u(0)=u(1)\}$, then, in addition to $(*)$, $u\neq 0$ must satisfy $u(0)=u(1)$, which is possible only for $\lambda=0$. Hence

$$\boxed{\sigma_{point}(A_2)=0}$$

If $D(A)=\{u\in H^1: u(0)=u(1)=1\}$, then no $u\neq 0$ can satisfy both $(*)$ and $u(0)=u(1)=0$. Hence

$$\boxed{\sigma_{point}(A_3)=\emptyset}.$$

(3) For $\varphi\in S(\mathbb{R})$,

$$\langle |x|',\varphi\rangle=-\langle |x|,\varphi'\rangle=-\int_{-\infty}^{\infty}|x|\varphi'(x)\,dx$$

$$=\int_{-\infty}^{0}x\varphi'(x)\,dx-\int_{0}^{\infty}x\varphi'(x)\,dx$$

$$=-\int_{-\infty}^{0}\varphi(x)\,dx+\int_{0}^{\infty}\varphi(x)\,dx \qquad \text{(by parts)}$$

$$=\langle \operatorname{sgn}(x),\varphi\rangle. \quad \text{where } \operatorname{sgn}(x)=\begin{cases}-1, & x<0\\ 1, & x>0\end{cases}$$

and the value of $\operatorname{sgn}(0)$ does not matter (can be defined in any way).

Hence $|x|'=\operatorname{sgn}(x)$ in the distributional sense.

$\operatorname{sgn}(x)$ is a regular distribution.

(4) (6) A is unbounded:

Indeed, let $h \in L^2$ be a function such that $x^2 h \notin L^2$ (for example, $h(x) = \frac{1}{1+x^2}$).

For a $R>0$, define $f_R = f \cdot \chi_{[-R,R]}$.

Then $\|f_R\|_{L^2} \le \|h\|_{L^2}$

but $\|Af_R\|_{L^2}^2 = \|x^2 f_R\|_{L^2}^2 = \int_{-R}^{R} |x^2 h|^2 dx \to \infty$ as $R \to \infty$

(because $x^2 h \notin L^2$)

This shows that A is unbounded.

A is formally self-adjoint:

Indeed, $\langle Af, g\rangle = \int x^2 f g \, dx = \langle f, Ag\rangle$.

$D(A^*) = D(A)$:

Recall that

$D(A^*) = \{ g \in L^2$: the linear functional $f \mapsto \langle Af, g\rangle$ is bounded on $D(A)\}$

(*) $= \{ g \in L^2 : \exists \; C>0$ such that $\left| \int x^2 f g \, dx \right| \le C \|f\|_{L^2}$ for all $f \in D(A)\}$

1) $D(A^*) \supset D(A)$ because for $g \in D(A)$,

$\left| \int x^2 f g \right| \le \|x^2 g\|_{L^2} \|f\|_{L^2}$ by Cauchy-Schwartz

so one can take $C = \|x^2 g\|_{L^2}$

2) $D(A^*) \subset D(A)$.

Indeed, let $g \in D(A^*)$ and define $f_R = x^2 g \cdot \chi_{[-R,R]}$

Then $f_R \in D(A)$ and $\left| \int x^2 f_R g \right| = \|f_R\|_{L^2}^2$

Hence, by (*), we must have for some constant C:

$\|f_R\|_{L^2}^2 \le C \|f_R\|_{L^2} \quad \forall R$

which is $\|f_R\|_{L^2} \le C \quad \forall R$.

This is true only if $x^2 g \in L^2$ (by the definition of f_R).

Hence $g \in D(A)$. QED.

(5) Denoting $q(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, we can write our equation as

$$u' + q * u = f.$$

Applying Fourier Transform, we find that

$$ik\hat{u} + \sqrt{2\pi}\,\hat{q}\,\hat{u} = \hat{f}.$$

Since $\hat{q}(k) = \frac{1}{\sqrt{2\pi}} e^{-k^2/2}$, we solve this equation for $\hat{u}$ as

$$\hat{u} = \frac{\hat{f}}{ik + e^{-k^2/2}}.$$

By the inversion formula,

$$\boxed{u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{f}(k)}{ik + e^{-k^2/2}} e^{ikx}\, dk}$$