201C FINAL EXAM June 11, 2005

Closed book, closed notes. You may use any standard theorem, provided you state it accurately.

1. (a) State Plancherel's Theorem.

(b) Compute the value of the integral

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx.$$

(Hint: consider the Fourier Transform of an indicator function of an interval).

(c) Prove that the sequence of functions $\frac{\sin(nx)}{\pi x}$, considered as regular distributions, converges as $n \to \infty$ to the delta function at zero (in $S^*(\mathbb{R})$).

2. For $k = 1, 2, 3$, let $A_k : \mathcal{D}(A_k) \subset L^2[0,1] \to L^2[0,1]$ be the first-order differential operators $A_k u = u'$ with domains

$$\mathcal{D}(A_1) = H^1((0,1)),$$
$$\mathcal{D}(A_2) = \{u \in H^1((0,1)) \mid u(0) = u(1)\},$$
$$\mathcal{D}(A_2) = \{u \in H^1((0,1)) \mid u(0) = u(1) = 0\}.$$

Compute the point spectrum of A_1, A_2, A_3.

3. Compute the distributional derivative of the function $f(x) = |x|$ on $\mathbb{R}$. Is this derivative a regular or a singular distribution?

4. (a) Give the definition of an (unbounded) self-adjoint linear operator A with a dense domain $D(A)$ in a Hilbert space.

(b) Consider the multiplication operator $(Af)(x) = x^2 f(x)$ on $L^2(\mathbb{R})$ with the domain $D(A) = \{f \in L^2(\mathbb{R}) \mid x^2 f \in L^2(\mathbb{R})\}$. Prove that the operator A is unbounded and self-adjoint.

5. Solve the integro-differential equation (that is, give an expression for u in terms of f):

$$u'(x) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2}} u(y)\, dy = f(x).$$