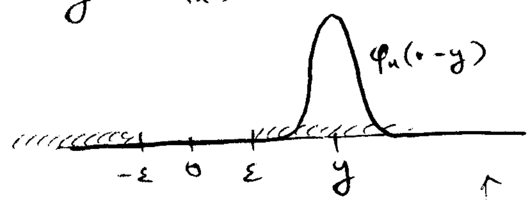
(1 B) Assume $p.v.\frac{1}{x}$ is a regular distribution, and let f be a function which defines it, that is

$$(*) \qquad \left(p.v.\tfrac{1}{x}\right)(\varphi) = \lim_{\varepsilon \to 0^+} \int_{|x|>\varepsilon} \frac{1}{x}\,\varphi(x)\,dx = \int_{\mathbb{R}} f(x)\,\varphi(x)\,dx \qquad \forall \varphi \in S(\mathbb{R}).$$

We shall prove that $f(x) = \frac{1}{x}$.

Applying $(*)$ to an approximate identity $\{\varphi_n,\ n=1,2\ldots\}$ we obtain for a fixed $y \neq 0$:



$$\left(p.v.\tfrac{1}{x}\right)(\varphi_n(\cdot - y)) = \lim_{\varepsilon \to 0^+} \int_{|x|>\varepsilon} \frac{1}{x}\,\varphi_n(x-y)\,dx$$

$$= \int_{\mathbb{R}} \frac{1}{x}\,\varphi_n(x-y)\,dx \quad \text{for all sufficiently large } n$$

(when $\operatorname{supp}[\varphi_n(\cdot - y)] \cap [-\varepsilon, \varepsilon] = \emptyset$)

$$= \underbrace{\left(\varphi_n * \frac{1}{x}\right)}_{\text{mollifier of } \frac{1}{x}}(y) = \underbrace{\left(\varphi_n * \left(\frac{1}{x}\cdot 1\!\!1_{[-M,M]}\right)\right)}_{\text{mollifier of } \frac{1}{x}\cdot 1\!\!1_{[-M,M]}}(y)$$

where M is a big number (such that $y \in (-M, M)$)

As we know (Thm. 10.16), the mollifiers converge in $L^2(\mathbb{R})$ to the original function, that is to $\frac{1}{y}\cdot 1\!\!1_{[-M,M]}$.

The same argument applies to the right hand side of $(*)$:

$$\int f(x)\,\varphi_n(x-y)\,dx \longrightarrow f(y) \quad \text{in } L^2(\mathbb{R})$$

This means (equate):

$$f(y) = \frac{1}{y}.$$

But $\frac{1}{y}$ does not define a regular distribution (nonintegrable singularity at 0). Q.E.D.

(2c) By the definition of the derivative, $\forall\, \psi \in S(\mathbb{R})$

$$\langle \varphi \delta', \psi \rangle = \langle \delta', \varphi \psi \rangle = -\langle \delta, (\varphi\psi)' \rangle$$

$$= -\langle \delta, \varphi'\psi + \varphi\psi' \rangle = -\varphi'(0)\psi(0) - \varphi(0)\psi'(0)$$

$$= -\varphi'(0)\langle \delta, \psi \rangle - \varphi(0)\langle \delta, \psi' \rangle$$

$$\cancel{= \varphi(0)\langle \psi, \delta'}$$

$$= \varphi(0)\langle \delta', \psi \rangle - \varphi'(0)\langle \delta, \psi \rangle$$

$$= \langle \varphi(0)\delta' - \varphi'(0)\delta, \psi \rangle.$$

Thus $\varphi\delta' = \varphi(0)\delta' - \varphi'(0)\delta.$

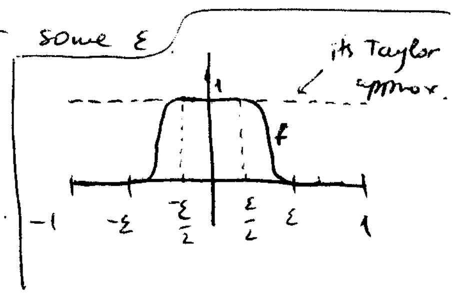
(3) a)

Let $f \in C^\infty([-1,1])$ be such that

$$f(0) = 1, \quad f^{(k)}(0) = 0 \ \forall\, k \geq 1$$

and $\mathrm{supp}(f) \subset (-\varepsilon, \varepsilon)$ for some ε.

Such functions exist for any $\varepsilon > 0$:



Then the Taylor series for f is identically 1.

$$Af = 1 \quad (\Rightarrow \|Af\|_{L^2} = 1)$$

while $\|f\|_{L^2} < \sqrt{2\varepsilon}$.

As $\varepsilon \to 0$, this shows that A is not bounded.

b) $D(A^*) = \{ g \in L^2$: the linear functional $f \mapsto \langle g, Af \rangle$ is bounded on $D(A)\}$.

"Bounded" means: $|\langle g, Af \rangle| \leq \|f\|_{L^2} \quad \forall\, f \in C^\infty$.

As in (a), we can construct functions with arbitrarily small $\|f\|_{L^2}$ and s.t. Af equals any monomial $(1, x, x^2, \dots, x^N)$.

Thus $\langle g, \forall \text{monomial} \rangle = 0$. Then $g \in (1, x, x^2, \dots, x^N)^\perp$.

So $\boxed{D(A^*) = (1, x, x^2, \dots, x^N)^\perp}$ Since on $D(A^*)$, $\langle A^* g, f \rangle = \langle g, Af \rangle = 0$, it follows that $\boxed{A^* = 0}$