201C MIDTERM EXAM May 13, 2005

Closed book, closed notes. You may use any standard theorem, provided you state it accurately.

1. (a) Give the definition of a tempered distribution and of a regular tempered distribution.

(b) Prove that the tempered distribution p.v.$\frac{1}{x}$ given by

$$\Big(\text{p.v.}\frac{1}{x}\Big)(\varphi) = \lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\, dx$$

is a singular distribution.

2. (a) Give the definition of the product φT of a test function $\varphi \in \mathcal{S}(\mathbb{R})$ and a tempered distribution T.

(b) Give the definition of the derivative of a tempered distribution T.

(c) Prove the identity

$$\varphi\delta' = \varphi(0)\delta' - \varphi'(0)\delta.$$

(d) Compute the derivative of the function $\log|x|$ as a distribution.

3. (a) Define the linear operator A on $L^2([-1,1])$ to be the N-th partial sum of the Taylor series at 0:

$$Af = \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!}\, x^k,$$

with the domain $D(A) = C^\infty([-1,1])$. Prove that the linear operator A is ubounded. *(Hint: find functions f arbitrarily small in L^2 sense, and for which Af are the monomials of degrees up to N.)*

(b) Calculate A^*.