Math 21A
Fall 2005

Midterm 1

Note : You must show *all* work to receive full credit on a problem. This test is closed-book and closed-note. You may not use a calculator.

Name: KEY

Student ID Number: ____________

Section Number: ______________

Problem(s)	Score
1 a	/ 9
1 b	/ 9
1 c	/ 9
1 d	/ 9
1 e	/ 9
1 f	/ 9
2	/ 16
3 a	/ 7
3 b	/ 7
4	/ 7
5	/ 9
Extra Credit	/ 5
Total	/ 100

1. (9 points each) Find the following limits. If the limit does not exist, write "DNE":

a) $\lim\limits_{x\to 3} \dfrac{x-3}{x^2-2x-3}$

$$\lim_{x\to 3} \frac{x-3}{x^2-2x-3} = \lim_{x\to 3} \frac{\cancel{(x-3)}}{(x+1)\cancel{(x-3)}} = \lim_{x\to 3} \frac{1}{x+1} = \boxed{\frac{1}{4}}$$

(by substitution into continuous function)

↑ denom. has roots 1, -3

b) $\lim\limits_{t\to 0} \dfrac{\sqrt{t+1}-1}{t}$

$$= \lim_{t\to 0} \frac{(\sqrt{t+1}-1)(\sqrt{t+1}+1)}{t(\sqrt{t+1}+1)}$$

(Multiply both numerator and denominator by $(\sqrt{t+1}+1)$)

$$= \lim_{t\to 0} \frac{(t+1)-1}{t(\sqrt{t+1}+1)}$$

$((a-b)(a+b) = a^2-b^2)$

$$= \lim_{t\to 0} \frac{\cancel{t}}{\cancel{t}(\sqrt{t+1}+1)} = \lim_{t\to 0} \frac{1}{\sqrt{t+1}+1} = \frac{1}{\sqrt{0+1}+1} = \boxed{\frac{1}{2}}$$

(by substitution into continuous function)

c) $\lim\limits_{x\to 0} \cos\left(\frac{1}{x}\right)$

DNE

d) $\lim\limits_{x\to 1^+} \dfrac{2-x^2}{1-x}$

$\lim\limits_{x\to 1^+} (2-x^2) = 2$, $\lim\limits_{x\to 1^+} (1-x) = 0$, $(1-x) < 0$ as $x \to 1^+$

Hence $\lim\limits_{x\to 1} \dfrac{2-x^2}{1-x} = -\infty$.

e) $\lim\limits_{x\to\infty} \sin x$

DNE

f) $\lim\limits_{x\to\infty} \dfrac{3x^2-1}{x^2-x-e^{-x}}$

$$= \lim_{x\to\infty} \frac{3-\frac{1}{x^2}}{1-\frac{1}{x}-\frac{1}{x^2e^x}}$$

(multiply numerator and denominator by $1/x^2$)

$$= \frac{\overset{3}{\overset{\parallel}{\lim\limits_{x\to\infty}(3)}} - \overset{0}{\overset{\parallel}{\lim\limits_{x\to\infty}\left(\frac{1}{x^2}\right)}}}{\underset{\underset{1}{\parallel}}{\lim\limits_{x\to\infty}(1)} - \underset{\underset{0}{\parallel}}{\lim\limits_{x\to\infty}\left(\frac{1}{x}\right)} - \underbrace{\lim\limits_{x\to\infty}\left(\frac{1}{x^2e^x}\right)}_{\parallel}}$$

$$\left(\underset{\underset{0}{\parallel}}{\lim_{x\to\infty} \frac{1}{x^2}}\right) \underset{\underset{0}{\parallel}}{\lim_{x\to\infty}\left(\frac{1}{e^x}\right)}$$

$$= \frac{3-0}{1-0-0} = \boxed{3}$$

2. (16 points) Give an ε, δ proof of the following limit: $\lim_{x \to 0} x \cos\left(\frac{1}{x}\right) = 0.$

We have to prove:

"For an arbitrary $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x| < \delta$ implies $|x \cos(\frac{1}{x})| < \varepsilon$."

Note that $|\cos(\frac{1}{x})| \leq 1$ for all x,

thus $|x \cos(\frac{1}{x})| \leq |x|$.

So, given an arbitrary $\varepsilon > 0$, we can choose $\delta = \varepsilon$ to conclude that:

$|x| < \delta$ implies $|x \cos(\frac{1}{x})| \leq |x| < \varepsilon$.

The claim is proved.

3. (7 points each) At what points are the following functions continuous? Explain.

a) $f(x) = \dfrac{\sin(x - \pi)}{x - \pi}$

The domain of f is $(-\infty, \pi) \cup (\pi, +\infty)$.

Both numerator and denominator are continuous on this domain, and denominator $\neq 0$. So, the function is continuous on its domain

$\boxed{(-\infty, \pi) \cup (\pi, +\infty)}$

b) $g(x) = \dfrac{\sqrt{x-1}}{1 + \cos^2 x}$

The denominator is never equal to 0, because $1+\cos^2 x \geq 1$, and it is continuous everywhere. The domain of the numerator is $[1, \infty)$ and it is continuous on its domain

Thus g is continuous on $[1, \infty)$

4. (7 points) At t sec after liftoff, the height of a rocket is $4t^2$ ft. How fast is the rocket climbing 10 sec after liftoff?

$h(t) = 4t^2$ is the height.

The velocity at time t is $v(t) = h'(t) = 8t$.

~~At~~ 10 sec after the liftoff, the velocity is

$$v(10) = \boxed{80 \text{ ft/sec}}$$

5. (9 points) Find an equation for the line tangent to the graph of the function $f(x) = 2\sqrt{x-1}$ at the point $(2, 2)$.

The equation of the tangent line to the graph of the function $y=f(x)$ at point (x_0, y_0) is

$$y = y_0 + f'(x_0)(x - x_0).$$

In this problem, $f(x) = 2\sqrt{x-1}$

$$f'(x) = \left[2(x-1)^{1/2}\right]' = (2 \cdot \tfrac{1}{2})(x-1)^{-1/2} = \frac{1}{\sqrt{x-1}}.$$

$x_0 = 2, \quad y_0 = 2,$

$$f'(x_0) = \frac{1}{\sqrt{2-1}} = 1$$

Thus the equation of the tangent line is

$$y = 2 + 1(x-2),$$

or

$$\boxed{y = x}.$$

7. (EXTRA CREDIT: 5 points) Show that the equation $x^3 - x - 1 = 0$ has a solution in the interval $[1, 2]$.

$f(x) = x^3 - x - 1$ is a continuous function

$f(1) = -1 < 0$,

$f(2) = 5 > 0$.

Thus, by the Intermediate Value Theorem, $f(x)$ takes on every value between -1 and 5 on the interval $[1, 2]$.

In particular, it takes on value 0 at some point x_0 in $[1, 2]$.

Obviously, $x_0 \neq 1$ and $x_0 \neq 2$ because $f(1), f(2) \neq 0$.

Thus $x_0 \in (1, 2)$.

The fact that $f(x_0) = 0$ means that

$$x_0^3 - x_0 - 1 = 0,$$

i.e. x_0 is a solution of the equation $x^3 - x - 1 = 0$.