MATH 21A-1 (Spring 2005) Name: Exam Key

Isaiah Lankham **Midterm #1** Date: April 22, 2005

You do not need to show any of your work or justify your reasoning for the problems below.

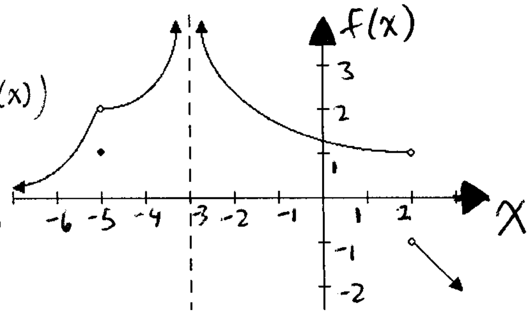
Problem 1. (Two points each) For each of the following statements, **circle the appropriate answer** to indicate whether the statement is true or false.

(a) **True** or **False**:
The following is the <u>definition</u> of a function $f(x)$ being continuous on the closed interval $[a, b]$:

If m is any value between $f(a)$ and $f(b)$, then there is some $c \in [a, b]$ such that $f(c) = m$.

(This is the Intermediate Value Theorem for cts fcns on a closed interval.)

For parts (b)–(e), refer to the graph of the function $f(x)$ given in the figure below:



(b) **True** or **False**: $\lim_{x \to -5} f(x)$ exists. (and is 2)

(c) **True** or **False**: $\lim_{x \to 2} f(x)$ exists. $\left(\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)\right)$

(d) **True** or **False**: $f(x)$ is differentiable at $x = -3$.
(It's certainly not even cts there!)

(e) **True** or **False**: $f(x)$ has four points of discontinuity shown in the picture.
(There are only 3 such points @ $x = -5, -3$, and 2.)

(f) **True** or **False**:
If $f(x)$ is a <u>continuous</u> function on the open interval (a, b), then the two-sided limit $\lim_{x \to c} f(x) = f(c)$ for any point $c \in (a, b)$. (This is by the def'n of a fcn being cts.)

(g) **True** or **False**:
If $f(x)$ is a <u>continuous</u> function on the open interval (a, b), then $f(x)$ necessarily <u>attains a maximum value</u> on (a, b). (This is only necessarily true on a <u>closed</u> interval.)

(h) **True** or **False**:
If $f(x)$ is a <u>differentiable</u> function on the open interval (a, b), then $f(x)$ is necessarily <u>continuous</u> on (a, b). (This is a basic fact about differentiable fcns discussed in class.)

(i) **True** or **False**:
If $f(x)$ is a <u>differentiable</u> function on the closed interval $[a, b]$, then $f(x)$ necessarily <u>attains a minimum value</u> on $[a, b]$. (By (h) and the Intermediate Value Th'm.)

(j) **True** or **False**:
The precise definition of the limit (i.e., the one involving ϵ's and δ's) was developed by Karl Weierstrass in 1859 for the sole purpose of torturing students in a Calculus class like this one.
(At least we can assume that this is false...)

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Problem 2. (Five points each) For each of the following questions, <u>**circle all correct answers**</u> among those provided.

(a) The following expressions involving the symbols ∞ and 0 are <u>indeterminate</u>:

(I) ∞/∞

(II) $\infty/0$ $= \infty \cdot \frac{1}{0} = \infty \cdot \infty = \infty$

(III) $\infty - \infty$

(IV) $0/0$

(V) $\infty \cdot \infty$ $= \infty$

(b) The domain of the function $f(x) = \sqrt{16 - x^2}$ <u>includes</u> the following points:

(I) $x = 0$ $(f(0) = 4)$

(II) $x = -4$ $(f(-4) = 0)$

(III) $x = -2$ $(f(-2) = \sqrt{12})$

(IV) $x = \sqrt{16}$ $(f(\sqrt{16}) = 0)$

(V) $x = 16$ ($f(16)$ is undefined)

(c) Given only that $\lim_{x \to 2} f(x) = 3$ and $\lim_{x \to 2} g(x) = 5$, the following limits <u>necessarily exist</u>:

(I) $\lim_{x \to 2} (f(x) \cdot g(x))$ $= 3 \cdot 5 = 15$

(II) $\lim_{x \to 2} ((f(x) - 3) \cdot \cos(g(x)))$ $= 0 \cdot \cos(5) = 0$

(III) $\lim_{x \to 2} \frac{f(x)-3}{g(x)-5}$ $= \frac{0}{0}$ (Indeterminant)

(IV) $\lim_{x \to 2} \frac{1}{f(x)-3} \cdot \frac{1}{g(x)-5}$ $= ?$ (Looks like $\infty \cdot \infty$, but we can't break it up!)

(V) $\lim_{x \to 15} (f(x) + g(x))$ $= ?$ (We only have information about limits at $x = 2$.)

(d) If $f(x) = \frac{x^2}{x^3 - 1}$, then the following are <u>necessarily true</u>:

(I) $f(x)$ has a removable discontinuity at $x = 1$. (It's an "infinite discontinuity b/c of the "blow-up")

(II) $f(x)$ has the horizontal asymptote $y = 1$. (The horizontal asymptote is @ $y = 0$.)

(III) $f(x)$ has the slant asymptote $y = x$ (Try long division.)

(IV) $f(x)$ is continuous on the interval $[2, 4]$. (There are no "blow-ups," etc.)

(V) $f(x) > 0$ or has a "blow-up" (i.e., goes to $\pm\infty$) for all $x > 0$. (Try $x = \frac{1}{2}$.)

Extra Credit. (Three points) In class a mnemonic device was given for remembering the first ten digits in the decimal expansion of the number $e = 2.71828\ldots$. What was the exact mnemonic?

2.7 (Andrew Jackson)(Andrew Jackson)

Andrew Jackson served 2 terms as the 7th US President and was elected in 1828.

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Problem 3. (One point per blank) **Fill in the blanks** in each of the following statements:

(a) Given $f(x) = \frac{3x}{x^2-3}$, the domain of $f(x)$ cannot contain $x =$ $\pm\sqrt{3}$ because these point(s) cause the function to "blow-up" — i.e., to tend to infinity s.t. $f(\pm\sqrt{3})$ is undefined.
In interval notation, the domain of $f(x)$ is $\mathbb{R}\setminus\{\pm\sqrt{3}\} = (-\infty,-\sqrt{3})\cup(-\sqrt{3},\sqrt{3})\cup(\sqrt{3},\infty)$.

(b) Suppose we have a rectangle with side lengths x and y. Then the area $A =$ $x \cdot y$ and the perimeter $P =$ $2x+2y$. Solving for y in the latter expression, $y =$ $\frac{1}{2}(P-2x)$.

Substituting this into the expression for A, we can express the area as a function of the side length x as follows: $A(x) =$ $\frac{1}{2}x(P-2x)$, which has domain $(0,\infty)$ or $(0, \frac{1}{2}P)$.
(depending on the application)

(c) If $f(x) = x^2 - x - 1$ and $g(x) = \cos(x)$, then (without simplifying)

(i) $f(x+h) - g(x-h) =$ $((x+h)^2-(x+h)-1)-(\cos(x-h))$.

(ii) $(f \circ g)(x) + f(2+h) =$ $(\cos x)^2-(\cos x)-1 + (2+h)^2-(2+h)-1$.

(iii) $(g \circ f)(x) - g(x^2+h) =$ $\cos(x^2-x-1) + \cos(x^2+h)$.

(d) Let $f(x) = (1 + \frac{1}{x^2})$. Then the limit $\lim_{x\to+\infty} f(x) =$ 1 because given any $E >$ 0, we can find a $D =$ $E^{-\frac{1}{2}}$ > 0 such that if $x >$ D, then $|f(x)-1|$ $< E$.

(e) Suppose that the limit $\lim_{x\to a} f(x)$ exists. Then $\lim_{x\to a} f(x) = L$ means that for all $\varepsilon>0$, there exists a $\delta>0$ such that if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$.

Extra Credit. (Five points) The modern equals sign "=" is believed to have originated in a famous book called "The Whetstone of Witte", which was first published in 1557 and written by an amateur mathematician named Robert Recorde.

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Show all of your work and carefully explain your reasoning. Unclear answers will receive no credit.

Problem 4. (Five points each) Compute the following limits or state that they do not exist. Explicitly state when you are using any "u-substitutions" or when you are using any special limits.

(a) $\lim_{x\to-\infty}\left(\frac{3x^2+2x}{x+5}-3x\right) = \lim_{x\to-\infty}\frac{3x^2+2x-3x^2-15x}{x+5} = \lim_{x\to-\infty}\frac{-13x}{x+5} = \lim_{x\to-\infty}\frac{-13}{1+1/x} = \boxed{-13.}$

multiply both numerator and denominator by $\frac{1}{x}$.

(b) $\lim_{x\to 2}\frac{1+2x}{x-1} = \frac{1+2(2)}{2-1} = \boxed{5.}$

Just "plug in" $x=2$ since nothing "bad" happens.

(c) $\lim_{x\to 0}\frac{\sin^2(2x)}{12x} = \underbrace{\lim_{x\to 0}\frac{\sin(2x)}{2x}}\cdot\lim_{x\to 0}\frac{\sin(2x)}{6} = 1\cdot 0 = \boxed{0.}$

Special Limit: $\lim_{x\to 0}\frac{\sin(cx)}{cx}=1,\ \forall c\in\mathbb{R}$.

(d) $\lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0}\frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h)$

$= \boxed{2x.}$

(e) $\lim_{x\to 0}\sin(\frac{1}{x})$ $\boxed{\text{Does Not Exist.}}$

Set $u=\frac{1}{x^2}$ Special Limit: $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}=1$

(f) $\lim_{x\to+\infty}x^2\sin(\frac{1}{x^2}) = \lim_{u\to 0}\frac{\sin(u)}{u} = 1$

Problem 5. (Ten points total) Show that the equation $x^5+4x^4-6x^3-8x+2=2^x-\cos(\pi x)$ has a solution on the interval $[0,1]$, and explicitly state any theorem(s) you use.

Set $f(x)=x^5+4x^4-6x^3-8x+2-2^x+\cos(\pi x)$. Then since $f(0)=2>0$ and $f(1)=-10<0$, and since $f(x)$ is continuous on $[0,1]$, we can apply the Intermediate Value Th'm to conclude that $\exists\ c\in[0,1]$ s.t. $f(c)=0$

$\Rightarrow c^5+4c^4-6c^3-8c+2=2^c-\cos(\pi c)$.