MATH 21A-1 (Spring 2005) Name: Exam Key

Isaiah Lankham **Midterm #2** Date: May 13, 2005

You do not need to show any of your work or justify your reasoning for the problems below.

Problem 1. (Two points each) For each of the following statements, **circle the appropriate answer** to indicate whether the statement is true or false.

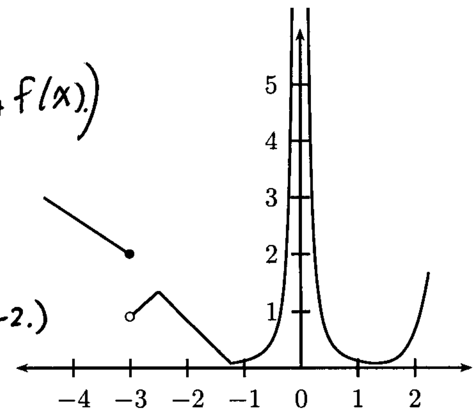
(a) **True** or **False**

The following is the definition of a function $f(x)$ being differentiable on the closed interval $[a, b]$:

There is some $c \in (a, b)$ such that the value of the derivative $f'(c)$ is equal to $\dfrac{f(b) - f(a)}{b - a}$.

(This is the property guaranted by the Mean Value Theorem.)

For parts (b)–(e), refer to the graph of the function $f(x)$ given in the figure below:



(b) **True** or **False** $\lim_{x \to -3} f(x)$ exists. $\left(\lim_{x \to -3^-} f(x) \neq \lim_{x \to -3^+} f(x)\right)$

(c) **True** or **False** $f'(x)$ exists for every $x \in (-2, -1)$.

(we can't take a tangent line near $x = -1$.)

(d) **True** or **False**: $f''(x) = 0$ at $x = -2$.

(since $f(x)$ is locally a straight line near $x = -2$.)

(e) **True** or **False** $f(x)$ has five points shown in the picture where it is not differentiable.

(There are only four such points.)

(f) **True** or **False**

If $f(x)$ is a function that is differentiable on the closed interval $[a, b]$, then the two-sided limit $\lim_{x \to c} f(x) = f(c)$ for any point $c \in [a, b]$.

(We can't necessarily take 2-sided limits at the endpoints $x = a, b$.)

(g) **True** or **False**

If $f(x)$ is a differentiable function on the closed interval $[a, b]$ and $f'(c) = 0$ for some point $c \in (a, b)$, then $f(x)$ necessarily has a relative minimum or maximum at $x = c$.

(Cf. $f(x) = x^3$, when $x = 0$.)

(h) **True** or **False**:

If $f(x)$ and $g(x)$ are two differentiable functions, then $(g \cdot f)'(x) = g(x) \cdot f'(x) + f(x) \cdot g'(x)$ by the product rule.

(This is just the statement of the Product Rule.)

(i) **True** or **False**

If $f(x)$ and $g(x)$ are two differentiable functions, then $(g \circ f)'(x) = g'(f'(x)) \cdot f(x)$ by the chain rule.

(The Chain Rule states that $(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$.)

(j) **True** or **False**

The Mean Value Theorem and its generalizations were developed by several prominent European Mathematicians (most notably Joseph Louis Lagrange and Augustin Louis Cauchy) around the turn of the nineteenth century in order to stress out American Calculus students 200 years later.

(They instead did some of the earliest work on the precise definition of the limit — the one with ε's and δ's — for this purpose ☺.)

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Problem 2. (Four points for (a); Two points for each response in part (b))

(a) State the Mean Value Theorem using proper mathematical notation:

If $f(x)$ is a continuous function on the closed interval $[a,b]$ that is also differentiable on the open interval (a,b), then $\exists\, c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.

(b) Several consequences of the Mean Value Theorem were given in class. State two of them:

(1) We can uniquely find antiderivatives on $[a,b]$. (Up to an additive constant, at least.)

(2) If a differentiable function is increasing (resp. decreasing), then its derivative is positive (resp. negative).

Problem 3. (One point per blank) **Fill in the blanks** in each of the following statements:

(a) The derivative $\frac{d}{dx}\sin(x) = \cos(x)$ because the difference quotient $\frac{\sin(x+h)-\sin(x)}{h}$ can be simplified using the trig identity $\sin(x+h) =$ $\sin(x)\cos(h) + \cos(x)\sin(h)$ so that

$$\frac{\sin(x+h)-\sin(x)}{h} = \sin(x)\cdot \frac{\cos(h)-1}{h} + \cos(x)\cdot \frac{\sin(h)}{h}$$

(b) Given a right circular cylinder with height h and radius r, the volume $V =$ $\pi r^2 h$ and the surface area $A =$ $2\pi r h + 2\pi r^2$. Solving for h in the latter expression, $h =$ $(A - 2\pi r^2)/(2\pi r)$. Substituting this into the expression for V, we can express the volume as a function of the radius r as follows:

$$V(r) = \pi r^2\left(\frac{A-2\pi r^2}{2\pi r}\right) = \frac{r}{2}(A - 2\pi r^2),$$ which has domain $r \in (0,\infty)$ or $r \in \left(0, \sqrt{\frac{A}{2\pi}}\right)$ (depending on the application)

(c) Suppose that the limit $\lim_{x\to a} f(x)$ exists. Then $\lim_{x\to a} f(x) = L$ means that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

Extra Credit. (Two points each) The third derivative of the position function is called the Jerk/Surge/Jolt and the fourth derivative of the position function is officially known as the Jounce (Snap is informal)

(Four points) A Russian stacking doll (the analogy given in class for the Chain Rule) is called a Matryoska Doll, with a female doll called a "baboushka" and a male doll a "dedoushka".

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Show all of your work and carefully explain your reasoning. Unclear answers will receive no credit.

Problem 4. (Five points each) Compute the following derivatives. **YOU DO NOT NEED TO SIMPLIFY**.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(x^3+15)(x^2+2)}{12x^7-1}\right) = \boxed{\frac{(12x^7-1)\overbrace{\left[(x^2+2)(3x^2)+(x^3+15)(2x)\right]}^{5x^4+6x^2+30x} - \overbrace{(x^3+15)(x^2+2)}^{x^5+2x^3+15x^2+30}(84x^6)}{(12x^7-1)^2}}.$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt[5]{(2x^3-x^2+1)^5+x^5}\right) = \boxed{\frac{1}{5}\left((2x^3-2x+1)^5+x^5\right)^{-\frac{4}{5}}\left[5(2x^3-x^2+1)^4(6x^2-2x)+5x^4\right]}.$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\,(12x^7-1)\cdot\cos(x^3+15)\,\right) = \boxed{-\sin(x^3+15)(3x^2)(12x^7-1)+(\cos(x^3+15))\cdot(84x^6)}.$

(d) $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\sec(x)) = \frac{d}{dx}(\sec(x)\cdot\tan(x)) = \boxed{(\sec^2(x))\cdot\sec(x)+(\sec(x)\tan(x))\cdot\tan(x)}.$

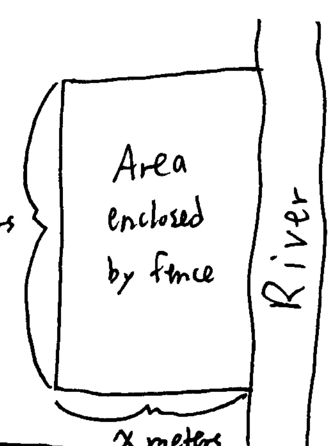
Problem 5. (Ten points) Solve the following optimization problem using Calculus and clearly explain your reasoning: A farmer named Paul wants to put a fence around three sides of a rectangular field, using a straight river as the fourth side. Given that Paul has exactly 1200 meters of fencing material available, what is the largest possible area that can be enclosed?

(1) Objective: Maximize Area $A = x\cdot y$ enclosed by fence in picture:

(2) Constraint: We're constrained by having a fixed perimeter $2x+y=1200$.

(3) Combine: Solving, $y = 1200-2x$, so we can express $A(x) = x(1200-2x) = 1200x-2x^2$.

(4) Optimize: Since $A'(x) = 1200-4x = 0$ when $x = 300$, we must have that the maximum area is $\boxed{A(300) = 180{,}000\, m^2}$.



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Show all of your work and carefully explain your reasoning. Unclear answers will receive no credit.

Problem 6. (Five points each) Let $f(x) = \dfrac{x+1}{x}$.

(a) Use the **definition of the derivative** to calculate $f'(x)$. (HINT: Try writing $f(x) = 1 + \frac{1}{x}$ first.)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left(1+\frac{1}{x+h}\right) - \left(1+\frac{1}{x}\right)}{h} = \lim_{h\to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{x-(x+h)}{h\cdot x\cdot(x+h)} = \lim_{h\to 0} \frac{-h}{h\cdot x\cdot (x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}. \text{ So, } \boxed{f'(x) = -x^{-2}}.$$

(b) Find the tangent line to $f(x)$ when $x = 1$.

Slope: $f'(1) = -(1)^{-2} = -1$

Point: $f(1) = \frac{1+1}{1} = 2$

$\Rightarrow$ Tangent Line @ $x=1$ has the equation $y-2 = -1\cdot(x-1)$ or $\boxed{y=-x+3}$.

(c) Prove or disprove: the tangent line found in part (b) passes through the point $(3, 0)$.

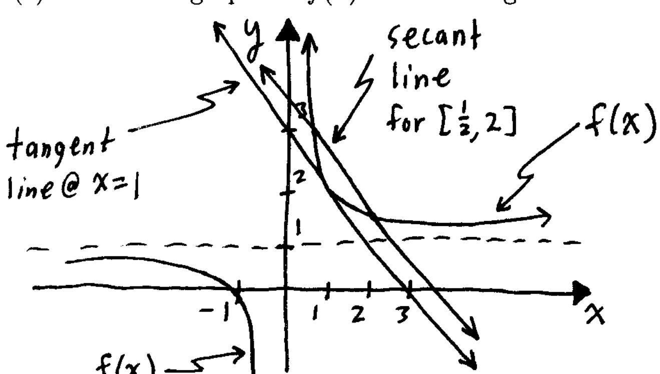
Writing the equation for the tangent line in part (b) as $y(x) = -x+3$, note that $y(3) = -3+3 = 0$ so that this is $\boxed{\text{true}}$.

(d) Find all values c satisfying the Mean Value Theorem for $f(x)$ on the interval $[\frac{1}{2}, 2]$.

Clearly $f(x)$ is differentiable on $[\frac{1}{2}, 2]$ so that we can apply the MVT to get that $\exists c \in (\frac{1}{2}, 2)$ s.t. $f'(c) = \dfrac{f(2) - f(\frac{1}{2})}{2 - \frac{1}{2}} = -1$.

I.e., $-c^{-2} = -1$ so that $c = \pm 1$, but then only $\boxed{c = 1 \in (\frac{1}{2}, 2)}$.

(e) Sketch the graphs of $f(x)$ and the tangent line found in part (b). Graphically verify part (d).



Note that the tangent line @ $x=1$ is parallel (i.e., has the same slope $m=-1$) as the secant line connecting $(\frac{1}{2}, f(\frac{1}{2}))$ and $(2, f(2))$.

(f) Recall the **Power Rule**: $(x^n)' = nx^{n-1}$. Can we find an antiderivative for $f(x)$ by just "reversing" the Power Rule? Does anything go wrong? (HINT: Try writing $f(x) = 1 + x^{-1}$ first.)

If we try "reversing" the Power Rule on $f(x)$, we get the "antiderivative" $x + \dfrac{x^{-1+1}}{-1+1} = x + \dfrac{x^0}{0}$ which makes no sense since we can't divide by zero! (We'll see that $x + \ln(x)$ is the correct antiderivative.)