## Problem Set 1

Fall 2006: 202, Functional Analysis. Due November 13, 2006

**1.** Contradiction to the Open Mapping Theorem? Consider normed spaces X = C[0,1] and  $Y = \{f \in C^1[0,1], f(0) = 0\}$ , both with the sup-norm. Consider the intergation operator  $A : X \to Y$  defined as

$$(Af)(t) = \int_0^t f(s) \ ds.$$

(a) Prove that A is a bounded linear operator, one-to-one, and onto. Prove that  $A^{-1}$  is unbounded. Does this contradict the Open Mapping Theorem?

(b) Now have a closer look at the proof of the Open Mapping Theorem. Let D denote the unit ball of X and F = A(D). Describe the functions in F. Prove that the interior of F is empty but the center (a.k.a. kernel) of F is non-empty. (Compute the center of F). Does this contradict Theorem 9.1.3?

(c) Strengthen the statement that F has empty interior by proving that F is nowhere dense.

**2.** Stability of Solutions of Linear Equations. Let X and Y be Banach spaces, and  $T: X \to Y$  be a bounded linear operator. Assume that the equation

Tx = b

has a unique solution for every right hand side  $b \in Y$ . Show that the solution is stable under perturbations of the right hand side b. More precisely, there exists a constant C such that if  $||b_1 - b_2|| \le \varepsilon$  then the corresponding solutions  $x_1$  and  $x_2$  satisfy  $||x_1 - x_2|| \le C\varepsilon$ .

**3.** From Closed Graph to Banach-Steinhaus. Banach-Steinhaus theorem can be proved in many ways. In the class, we deduced it from Baire Category Theorem. In the textbook, it is deduced from Theorem 9.1.3 (that the center and the interior coincide for perfectly convex sets). In this exercise, try to deduce Banach-Steinhaus theorem from the Closed Graph Theorem by the following scheme.

Let  $G \subset L(X, Y)$  be a pointwise bounded family of linear operators. We want to prove that

$$M := \sup_{T \in G} \|T\| = \sup_{T \in G, x \in B_X, f \in B_{Y^*}} |f(Tx)|$$

is finite. Consider the space  $\ell_{\infty}(G \times B_{Y^*})$  of all bounded functions on  $G \times B_{Y^*}$ , with the sup-norm. Show that this a Banach space. Define the linear operator  $U: X \to \ell_{\infty}(G \times B_{Y^*})$  by

$$(Ux)(T,f) = f(Tx).$$

Prove that U has closed graph, and is thus bounded. It remains to observe that ||U|| = M.

4. C[0,1] is not reflexive. Prove that C[0,1] is not reflexive. One way to do this is to construct a functional  $f \in C[0,1]^*$  that does not attain its norm. Alternatively, one can

construct a non-reflexive subspace of C[0, 1]. (Consider the closed linear span of a sequence of continuous functions with disjoint support. What is this space isometric to?)

## 5. Uniqueness of Supporting Functionals.

(a) Show that in Hilbert spaces, every vector has a unique supporting functional.

(b) Show that the uniqueness fails in the space C[0, 1].

(c) Show that is a vector in a Banach space has more than one supporting functional, then it has uncountably many.

**6.** Reflexivity Passes to Quotients. We have proved in class that closed subspaces of reflexive spaces are reflexive. Prove that quotient spaces of reflexive spaces are also reflexive.