Problem Set 2

Fall 2006: 202, Functional Analysis. Due December 15, 2006

1. (Banach Limit) Develop a meaningful concept of the limit for all bounded (not necessarily convergent) sequences.

Precisely, prove that for every $x = (x_n) \in \ell_\infty$ there exists a number $\mathrm{Lim}(x_n)$, which coinsides with the ordinary limit for the convergent sequences, and which satisfies:

- $\liminf(x_n) \le \mathrm{Lim}(x_n) \le \limsup(x_n)$;
- $\mathrm{Lim}(x_n + y_n) = \mathrm{Lim}(x_n) + \mathrm{Lim}(y_n)$; $\mathrm{Lim}(cx_n) = c\,\mathrm{Lim}(x_n)$.

Hint. The Banach limit Lim has to be a linear functional on ℓ_∞. Consider the subspace $L_0 \subset \ell_\infty$ of all Cesaro convergent sequences, i.e. $x \in L_0$ if $\lim \frac{1}{n}(x_1 + \dots x_n)$ exists (called Cesaro limit). Define the Banach limit first on L_0 as the Cesaro limit, and show that it is bounded by the sublinear functional $\limsup(x_n)$. Then extend Lim onto the whole ℓ_∞.

2. (Projections vs. Extensions) Let X_0 be a closed subspace of a Banach space X, the following are equivalent:

(i) there exists a projection $P \in L(X, X)$ onto X_0;

(ii) for every Banach space Y, every operator $T_0 \in L(X_0, Y)$ can be extended to an operator $T \in L(X, Y)$.

3. (Strict Separation Theorems) Recall that subsets A, B of a normed space X are said to be separated by a hyperplane if there exists $f \in X^*$ and a constant θ such that

$$f(a) \le \theta \le f(b) \qquad \text{for all } a \in A,\ b \in B. \tag{1}$$

We say that the separation is strict if the strict inequalities hold in (1).

The Separation Theorem proved in class states that every two disjoint convex sets, one of which has nonempty interior, can be separated by a hyperplane. Deduce from this the following strict separation theorems:

(a) Every two disjoint convex closed subsets, one of which is compact, can be strictly separated by a hyperplane;

(b) Every two disjoint convex open subsets can be strictly separated by a hyperplane.

4. (Mazur's Lemma) Let x_n be a sequence in a Banach space, which converges weakly to a vector x. Prove that

$$\{x\} = \bigcap_{n=1}^{\infty} \overline{\mathrm{conv}(x_k)_{k \ge n}}.$$

(The inclusion $\subseteq$ was proved in class).

5. (General spectrum) We know that the spectrum of every bounded linear operator is a nonempty compact subset in $\mathbb{C}$. (Boundedness of the spectrum was proved in class;

nonemptiness is Proposition 9.8 in the textbook [Hunter-Nachtergaele]; for closedness, see p.75 of the textbook Eidelman-Milman-Tsolomitis]).

Prove the converse statement: every nonempty compact subset of $\mathbb{C}$ is the spectrum of some linear operator.

Hint. Consider a diagonal operator $T \in L(\ell_2, \ell_2)$ which acts on the canonical vector basis (e_n) as $Te_n = \lambda_n e_n$, where $\lambda_n \in \mathbb{C}$.

In the next two exercises, we say that the operator has finite rank if its image is a finite dimensional subspace.

6. Images of Compact Operators. Let $T \in L(X, Y)$ be a compact operator. We proved in class that the image of $\lambda I - T$ is closed for all $\lambda \neq 0$. Show that for $\lambda = 0$, the image is not closed for nontrivial operators. More precisely, prove that the image of a compact operator T is closed if and only if T has finite rank.

7. Compact vs. Finite Rank Operators. Consider the integral operator $T : C[0,1] \to C[0,1]$ defined as

$$(Tf)(\tau) = \int_0^1 K(t,\tau) f(t)\, dt \tag{2}$$

for some kernel $K(t,\tau) \in C([0,1]^2)$. Then T is a compact operator (Exercise 5.10 in the textbook [Hunter-Nachtergaele]). Prove that:

(a) If the kernel can be expressed in the form

$$K(t,\tau) = \sum_{k=1}^{n} \phi_k(t) \psi_k(\tau),$$

then T has finite rank.

(b) Every integral operator T as in (2) can be approximated with arbitrary accuracy (in the operator norm) by integral operators with finite rank.

Remark. Part (b) motivates a general question. Can one approximate every compact operator by operators of finite rank? This had been a longstanding open problem, solved in negative in the 70's. Yet, the answer is positive in spaces with (Shauder) bases, or more generally, in spaces with an approximation property. Most known spaces, including $C[0,1]$, have bases.