Non-Asymptotic Theory of Random Matrices
Lecture 14: SECTIONS OF CONVEX SETS VIA ENTROPY AND VOLUME II

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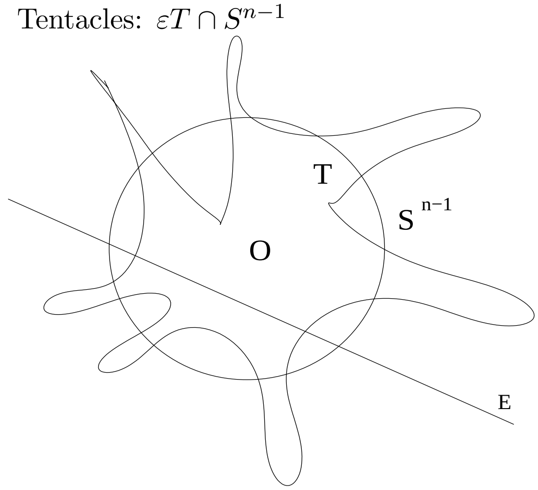
Tuesday, February 20, 2007

Theorem 1 (**Entropy Theorem**). *Let set $T \in \mathbb{R}^n$ be convex, covering number $N(T, B_2^n) \le V^n$, where V is the volume ratio - see Lecture 13; consider E: random subspace of $\mathbb{R}^n$ of codimension δn. Then :*

$$diam(T \cap E) \le C(V, \delta)$$

Tentacles: $\varepsilon T \cap S^{n-1}$



Realize $E = \ker G$, where $G : k \times n$ - Gaussian matrix ($\mathbb{R}^n \to \mathbb{R}^k$) ($k = \delta n$)

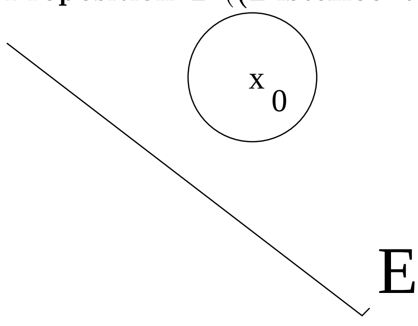
gaussian vector
↓

1. ($\dim n$) In $\dim n$ the inequality holds : $\mathbb{P}(||g - v||_2 < \varepsilon\sqrt{n}) \le (C'\varepsilon)^n$ for all $\varepsilon > 0$, all $v \in \mathbb{R}^n$ (this was proved before)
(volume of ε-ball $\sim \varepsilon^n$, tentacles do not change the order of value of the

volume of the set)

2. Replace gaussian vector g by E (let the above vector $v = 0$)

Proposition 2 (**(Distance to a subspace)**: Very sharp form [1]).



Let E be random subspace of $\mathbb{R}^n$ of codim k
Let $x_0 \in S^{n-1}$. Then :

(1) $(\mathbb{E}\ dist(x_0, E)^2)^{1/2} = \sqrt{\frac{k}{n}}$
(should interpolate between cases $k = 0$ (dist. $\sim \frac{1}{\sqrt{n}}$) and $k = n - 1$ (dist. ~ 1))
(2) $\mathbb{P}(dist(x_0, E) < \varepsilon\sqrt{\frac{k}{n}}) \leq (C\varepsilon)^k = C^k \varepsilon' \sqrt{\frac{n}{k}}$ *for* $\varepsilon > 0$

Proof: (1) *Exercise*
(2) (Weaker estimate) Let $E = KerG$ (G - a random map $\mathbb{R}^n \to \mathbb{R}^k$), then

$$dist(x_0, KerG) = \inf_{x \in KerG} ||x_0 - x||_2 \leq \inf_{x \in KerG} \frac{||G(x_0 - x)||_2}{||G||} = \frac{||Gx_0||_2}{||G||} \leq ?$$

Estimate numerator and denominator of the last fraction:

(a) for $A := \{||G|| \leq 2\sqrt{n}\}$, we have $\mathbb{P}(A) \leq 1 - e^{-cn}$ (see Lecture 6)
(b) Gx_0 is a standard Gaussian vector in $\mathbb{R}^k$ (x_0 - unit vector)

By $(\dim n)$ inequality above,

$$\mathbb{P}(||Gx_0|| < \varepsilon\sqrt{k}) \leq (C'\varepsilon)^k \qquad (*)$$

(if ε is small enough, $(C\varepsilon)^k$ may be $< e^{-cn}$).

Consider $\mathbb{P}_A$ = probability conditional on A :

$$\mathbb{P}_A(B) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)},$$

$$\mathbb{P}_A \left(\frac{||Gx_0||_2}{||G||} < \varepsilon \sqrt{\frac{k}{n}} \right) \leq \mathbb{P}_A \left(||Gx_0||_2 < \frac{\varepsilon}{2}\sqrt{k} \right) \leq \left(C' \frac{\varepsilon}{2} \right)^k \qquad \textit{by } (*)$$

We proved the result for $\mathbb{P}_A$ rather than $\mathbb{P}$.
But A is "big" event ($\mathbb{P}(A) \underset{n\to\infty}{\longrightarrow} 1$).

Proof of Entropy Theorem:

Goal: $T \cap E \leq \frac{1}{\varepsilon} B_2^n$ for $\varepsilon = \varepsilon(V, \delta) \iff \varepsilon T \cap E \leq B_2^n$ - open ball, or

$$(\varepsilon T \cap S^{n-1}) \cap E = \varnothing$$

(here $\varepsilon > 0$ - parameter)
Discretize the tentacles:
Choose an ε-net $\mathcal{N}$ of $\varepsilon T \cap S^{n-1}$, then the cardinality

$$|\mathcal{N}| = N(\varepsilon T \cap E, \varepsilon B_2^n) \ \leq\ N(\varepsilon T, \varepsilon B_2^n) \ =\ N(T, B_2^n) \ \leq\ V^n$$

- Fix an $x_0 \in \mathcal{N}$. The subspace E is far from x_0 with high probability:

$$\mathbb{P}_A \left(Ball(x_0, \varepsilon) \cap E \neq \varnothing \right) \ \leq\ \left(C\varepsilon \sqrt{\frac{n}{k}} \right)^k \ =\ \left(\frac{C\varepsilon}{\sqrt{\delta}} \right)^k$$

- Union bound: E is far from all $x_0 \in \mathcal{N}$:

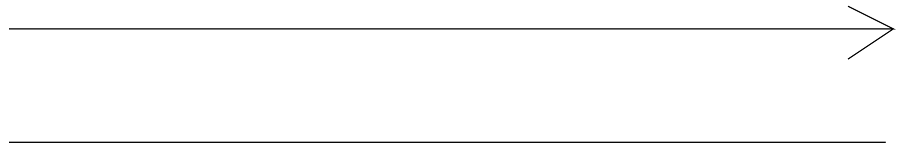
$$\mathbb{P}_A \left(\forall x_0 \epsilon \mathcal{N}, Ball(x_0, \varepsilon) \cap E = \varnothing \right) \ >\ 1 - |\mathcal{N}| \left(\frac{C\varepsilon}{\sqrt{\delta}} \right)^k \ =\ 1 - V^n \left(\frac{C\varepsilon}{\sqrt{\delta}} \right)^{\delta n} \ \geq\ 1 - e^{-n}$$

for suitable $\varepsilon = \varepsilon(V, \delta)$
Since $\mathbb{P}(A) > 1 - e^{-cn}$ it follows that

$$\mathbb{P} \left((\varepsilon T \cap S^{n-1}) \cap E = \varnothing \right) > 1 - e^{-cn}$$

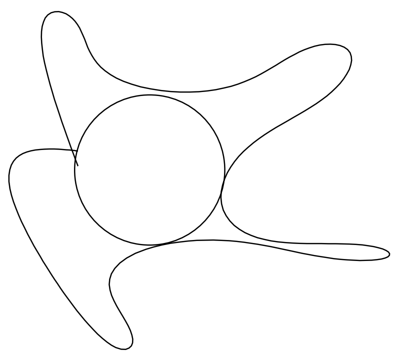
(check !)

Ex.: "Sausage" :



We have

Theorem 3 (**Volume Ratio Theorem**: [5], Ch.6)**.** *Let* $T \in \mathbb{R}^n$, $B_2^n \in T$

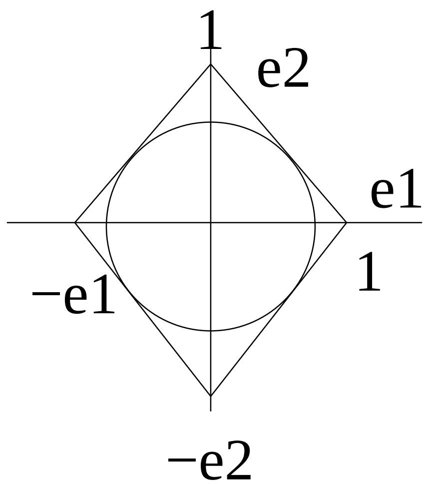


$$V(T) = \left(\frac{\mathsf{Vol}(T)}{\mathsf{Vol}(B_2^n)} \right)^{1/n}$$

$$diam(T \cap E) \leq C(V(T), \delta)$$

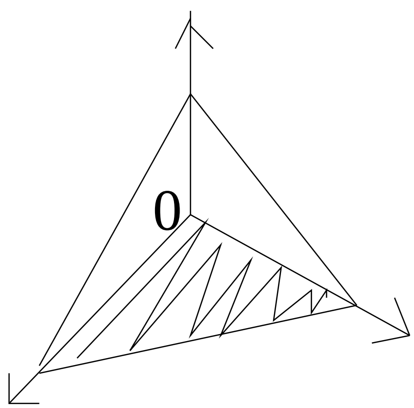
For the set T with tentacles above one has
Inscribed ball: radius 1.
Circumcribed ball: radius $C(V(T, \delta)$, hopefully $O(1)$
$\Rightarrow T \cap E$ is "almost spherical".

Let us consider balls for various norms of n-dimensional space.
Ex.: $B_1^n = Ball(l_1^n)$



Inscribed ball: $\frac{1}{\sqrt{n}}$
Circumscribed ball: 1

What is the volume $\mathsf{Vol}(B_1^n)\ = ?$
Ball $B_1^n = conv(\pm e_i)_1^n$ where e_i: coordinate basis, ($conv$ - convex hull)
So $\mathsf{Vol}(B_1^n) = 2^n \cdot \mathsf{Vol}(Simplex_n)$



where $Simplex_n = \{x \in \mathbb{R}^n;\ all\ x_i\ \geq\ 0,\ \sum x_i\ \leq\ 1\}$,

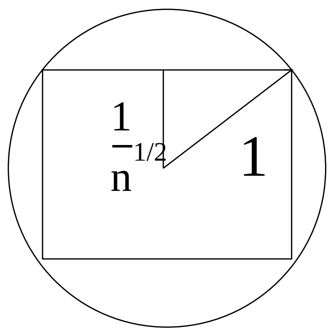
$$\mathsf{Vol}(Simplex_n) = \frac{1}{n}\mathsf{Vol}(Simplex_{n-1}) = \frac{1}{n!}$$

$$\Rightarrow \qquad \boxed{\mathsf{Vol}(\mathrm{B}_1^n) = \tfrac{2^n}{n!}}$$

We have also

$$\mathsf{Vol}(B_2^n) \geq \mathsf{Vol}\left(\frac{1}{\sqrt{n}} B_\infty^n\right) = \left(\frac{2}{\sqrt{n}}\right)^n$$

↑
(see [[5]])



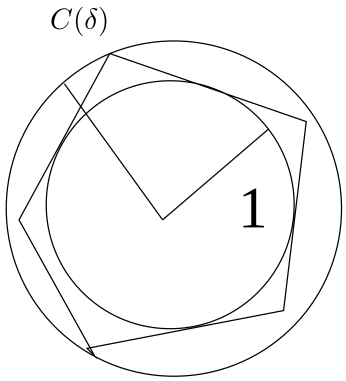
Apply Volume Ratio Theorem for $T = \sqrt{n} B_1^n$;
then $B_2^n \leq T$, $\mathsf{Vol}(T) = (\sqrt{n})^n \frac{2^n}{n!}$, and

$$V(T) = \left((\sqrt{n})^n \frac{2^n}{n!} \cdot \left(\frac{\sqrt{n}}{2}\right)^n\right)^{1/n} = \left(\frac{n^n}{n!}\right)^{1/n} \leq const$$

(use Stirling formula: $n! \approx n^n e^{-n} \sqrt{2\pi n}$)

Corollary 4 ([4])**.** *: For every $0 < \delta < 1$, a random subspace E of $\mathbb{R}^n$ of codimension δn satisfies with probability $1 - e^{-n}$:*

$$B_2^n \cap E \ \subseteq\ (\sqrt{n} B_1^n \cap E) \ \subseteq\ C(\delta) \cdot B_2^n \cap E$$



Equivalently, (let $c(\delta) = \frac{1}{C(\delta)}$),

$$c(\delta)||x||_2 \;\leq\; \frac{1}{\sqrt{n}}||x||_1 \;\leq\; ||x||_2 \qquad (*)$$

for all $x \in E$.

Corollary 5 (Kashin's Splitting [4]). *: $\exists$ an orthogonal decomposition*

$$\mathbb{R}^n = E \oplus F$$

into two $n/2$ - dimensional subspaces, s.t. (*) holds for both E and F.

Proof: Apply the corrollary to E and F (both are random, uniformly distributed).

1 Applications in Computer Science

Usually $||x||_1$ is easier to compute than $||x||_2$

Questions of applicability of [V. R. T.]: [5]
For subgaussian matrices (e.g. Bernoulli):
- should be true for $E = \ker G$ (try - where is the difficulty ?)
- true for $E = ImG$ [[7]]

• Dependence on δ ?
- In V.R.T. : must in general be exponential [[5]]
- For B_1^n : polynomial

$$C(\delta) \leq c\sqrt{\frac{1}{\delta}\log\frac{1}{\delta}}$$

(best possible estimate) [[2]] - gaussian case
- Subgaussian case in general - OPEN
- Bernoulli case - polynomial:

$$C(\delta) \leq \delta^{-5/2}\log\frac{1}{\delta}$$

(see [[6]])

Open problem: Explicit constructions of E in Kashin's Theorem (what is an appropriate basis ?)
Best known case: $\dim(E) = n^{1-\delta}$ (rather than $(1-\delta)n$)
See [P.Indyk: "Uncertainty Principles ..." [3]].

References

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[7] A. Litvak ; A. Pajor; M. Rudelson; N. Tomczak-Jaegermann. Smallest singular values of random matrices and geometry of random polytopes. *Advances in Mathematics*, 195:491, 2005.