Non-Asymptotic Theory of Random Matrices
Lecture 19: Small Ball Probability via Sum-Sets

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1 Small Ball Probability via Sum-Sets

Set

$$S = \sum_{k=1}^{n} a_k \xi_k$$

We want to bound the small ball probability,

$$P_\epsilon(a) = \sup_v \mathbb{P}(|s - v| \leq \epsilon) \leq ?$$

Theorem 1 (Esseen Inequality)**.**

$$\sup_v \mathbb{P}(|x - v| \leq 1) \lesssim \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\phi(t)| dt$$

where $\phi(t) = \mathbb{E}\mathrm{e}^{ixt}$ *is the characteristic function.*

Assumption 2. *Assume* $1 \leq a_k \leq k$ *for all* k*, and let* $\epsilon \leq \frac{\pi}{4}$*.*

By Esseen,

$$P_\epsilon(a) \lesssim \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\phi(\frac{t}{\epsilon})| dt$$

where

$$\phi(t) = \mathbb{E}\mathrm{e}^{ist} = \prod_{k=1}^{n} \phi_k(t)$$

and

$$\phi_k(t) = \mathbb{E}\mathrm{e}^{i a_k \xi_k t}$$

are the charactristic functions of $a_k \xi_k$.

Assumption 3. $\xi_k = \pm 1$ *(Bernoulli)*

In this case,

$$\phi_k(t) = \frac{1}{2}\left(\mathsf{e}^{ia_k t} + \mathsf{e}^{-ia_k t}\right) = \cos(a_k t)$$

$$P_\epsilon(a) \lesssim \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\prod_{k=1}^{n} \cos\left(\frac{a_k t}{\epsilon}\right)|dt.$$

The product over k is always less then or equal to one. It's equal to one when most of the $\cos\left(\frac{a_k t}{\epsilon}\right) \approx 1$, meaning $\frac{a_k t}{\epsilon}$ are approximately integer multiples of π.

1) Multiplication $\longrightarrow$ Additive: We first wish to replace the multiplication by addition. We have,

$$|x| \leq \exp(-1/2(1-x^2))$$

and so,

$$\begin{aligned} P_\epsilon(a) &\lesssim \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \prod_{k=1}^{n} \sin^2\left(\frac{a_k t}{\epsilon}\right)|dt \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathsf{e}^{-\frac{1}{2}f(\frac{t}{\epsilon})}dt \end{aligned}$$

where $f(t) = \sum_{k=1}^{n} \sin^2(a_k t)$.

Note that we want $f(t)$ to be large.

Example 4 (Ex)**.** *Set* $M := \max_{|t|\leq\frac{\pi}{2}} f(\frac{t}{\epsilon})$*. Then* $\frac{n}{4} \leq M \leq n$

Hint: Estimate the max by an average $\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \cdot\, dt$*.*

Think of $\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \cdot\, dt$ as an expectation with respect to t. Use the integral distribution formula,

$$\mathbb{E}x = \int_0^\infty \mathbb{P}(x \geq s)ds.$$

Then we have,

$$P_\epsilon(a) \leq \int_0^1 \left|\left\{t : |t| \leq \frac{\pi}{2} : \mathsf{e}^{-\frac{1}{2}f(\frac{t}{\epsilon})} \geq s\right\}\right| ds$$

Using the change of variables $s = \mathsf{e}^{-\frac{1}{2}m}$, we have

$$P_\epsilon(a) \leq \int_0^\infty \left| \underbrace{\left\{ t : |t| \leq \frac{\pi}{2} : f(\frac{t}{\epsilon}) \leq m \right\}}_{\textit{Level Sets}} \right| \cdot \mathsf{e}^{-\frac{m}{2}} dm$$

Definition 5 (Level Sets)**.** *Set*

$$T(m,r) = \left\{ T(m, \frac{\pi}{2})t : |t| \leq r, f(\frac{t}{\epsilon}) \leq m \right\}$$

Using this, we have that

$$P_\epsilon(a) \leq \int_0^\infty \left| T(m, \frac{\pi}{2}) \right| \cdot \mathsf{e}^{-\frac{m}{2}} dm,$$

which is a problem for small m, since $|T(m, \frac{\pi}{2})|$ decreases to zero as m decreases to zero. The regularity of decay of $|T(m, \frac{\pi}{2})|$ is approximatly $\sqrt{\frac{m}{M}}$.

2 Sum-Sets

See [?] on the Additive Combinatorics:
Note that here we use the Minkowski set addition:

$$A + B = \{a + b | a \in A, b \in B\}.$$

Theorem 6 (Mann-Kneser Inequality [?])**.** *On the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, For A,$B \subset \mathbb{T}$ closed, either $A + B = \mathbb{T}$ or $|A + B| \geq |A| + |B|$.*

Proof. Assume $A + B \neq \mathbb{T}$, and so take a boundary point $x \in A + B$. Then there exists an $x_j \to x$. We want to find disjoint translates of A and B in $A + B$. Then

$$(x_j - A) \cap B = \emptyset$$

Taking the limit,

$$|(x - A) \cap B| = 0.$$

Let $x = a + b$, with $a \in A$, $b \in B$. Then,

$$|(a + b - A) \cap B| = 0,$$

$$|(b - A) \cap (B - a)| = 0$$

and so we have almost disjoint translates of A,$B \subseteq B - A$ □

Corollary 7. *$T \subseteq \mathbb{T}$ closed, $T_N = \underbrace{T + ... + T}_{N \text{ times}}$*

Then either

$$T_N = \mathbb{T}$$

or

$$|T_N| \geq N \cdot |T|.$$

Also, for $\mathbb{R}$, $T \subseteq \mathbb{R}$ closed,

$$\mu_r(T) = |T \cap [-r, r]|$$

Then either

$$T \supseteq [-r, r]$$

or

$$\mu_r(T_N) \geq \frac{N}{2} \mu_{\frac{r}{2}}(T).$$

We want to apply sum-set bounds for $T := T(m, \frac{\pi}{2})$. Is there a z such that $T_N \subseteq T(z, N\frac{\pi}{2})$? We have

$$\sin(x + y) \leq |\sin x| + |\sin y|,$$

$$\sin\left(\sum_1^N x_j\right) \leq \sum_1^N |\sin x_j|,$$

$$\sin^2\left(\sum_1^N x_j\right) \leq N \sum_1^N \sin^2 x_j,$$

so

$$T_N \subseteq T(N^2 m, N \cdot \frac{\pi}{2})$$

On the other hand, we can use sum-set bound (Ex) to bound T_N.

Corollary 8 (Regularity)**.**

$$|T(N^2 m, \pi)| \geq \frac{N}{2} |T(m, \frac{\pi}{2})|$$

.

Note that rearranged, this corollary states that

$$|T(m, \frac{\pi}{2})| \leq \frac{2}{N} |T(\underbrace{N^2 m}_{\eta M}, \pi)|.$$

For every $\eta \in (0,1)$:

$$|T(m, \frac{\pi}{2})| \leq \sqrt{\frac{m}{\eta M}} \cdot |T(\eta m, \pi)|.$$

Thus

$$\begin{aligned} P_\epsilon(a) &\lesssim \int_0^\infty |T(m, \frac{\pi}{2})| \mathsf{e}^{-m/2} dm \\ &\lesssim \int_0^{\eta M} \sqrt{\frac{m}{\eta M}} \cdot |T(\eta M, \frac{\pi}{2})| \mathsf{e}^{-m/2} dm + \int_{\eta M}^\infty \pi \mathsf{e}^{-m/2} dm \\ &\leq \frac{1}{\sqrt{\eta M}} |T(\eta M, \frac{\pi}{2})| + \mathsf{e}^{-\eta M/2} \end{aligned}$$

And since $M \approx n$ from above, we have

$$P_\epsilon(a) \lesssim \frac{1}{\sqrt{\eta n}} |T(\eta n, \frac{\pi}{2})| + \mathsf{e}^{-cn/2}.$$

References

[1] A.M. Macbeath. *On measure of sum sets II, The sum-theorem for the torus*, volume 49. Cambridge, Cambridge, NY, 1953.

[2] Vu V. Tao, T. *Additive Combinatorics*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, NY, 2006.