Non-Asymptotic Theory of Random Matrices
Lecture 2: Concentration of Measure

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1 Concentration of the Volume

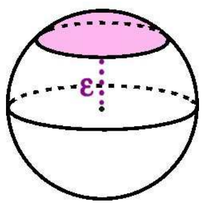
This lecture is based on results covered in [1, 2, 3]. Let $B_2^n = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$ denote the Euclidean ball in $\mathbb{R}^n$. For a constant $c > 0$, let cB_2^n be the scaled unit ball $\{cx : x \in \mathbb{R}^n, \|x\|_2 \leq 1\}$, and let $\mathsf{Vol}(A)$ denote the volume of A. Then we may ask questions about the relationships of different volumes. For instance, what is the relationship between $\mathsf{Vol}(B_2^n)$ and $\mathsf{Vol}(2B_2^n)$? Elementary geometry shows this relationship is precisely $\mathsf{Vol}(2B_2^n) = 2^n\mathsf{Vol}(B_2^n)$. Similarly, for $\epsilon > 0$, we have

$$\mathsf{Vol}((1+\epsilon)B_2^n) = (1+\epsilon)^n\mathsf{Vol}(B_2^n). \tag{1}$$

Using the fact that $(1+\frac{1}{n})^n$ tends toward a constant (e) as $n \to \infty$, we have that the volume of $(1+\epsilon)B_2^n$ and B_2^n are approximately equal if $\epsilon = O(\frac{1}{n})$. So this says that the sphereical shell of width $O(\frac{1}{n})$ contains most of the volume of the ball. Thus the volume of the ball is concentrated near the boundary, leading us to examine the concentration of the surface area.

2 Concentration of the Surface Area

Let $S^{n-1} = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ denote the Euclidean sphere in $\mathbb{R}^n$. Note that S^{n-1} is precisely the boundary of the Euclidean ball. Let σ denote the normalized rotationally invariant measure on S^{n-1} (so that $\sigma(S^{n-1}) = 1$). For $\epsilon > 0$ let $C(\epsilon)$ denote the spherical cap of height ϵ above the origin:

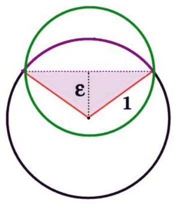


What is the area $\sigma(C(\epsilon))$? For a fixed $\epsilon > 0$, the area goes to 0 as $n \to \infty$. In fact, we have the following non-asymptotic result.

Lemma 1. *For $0 < \epsilon < 1$ the cap $C(\epsilon)$ of S^{n-1} satisfies*

$$\sigma(C(\epsilon)) \leq \mathsf{e}^{-n\epsilon^2/2}. \tag{2}$$

Proof: We first fix n and $\epsilon > 0$. We construct a cone that connects all points of $C(\epsilon)$ to the origin. In the figure below, this cone is shaded. Denote this cone by T. We then enclose the cone T in a ball with center ϵ above the origin. Using the Pythagoream Theorem, this ball can be chosen so that the radius is $\sqrt{1-\epsilon^2}$.



We then have by elementary geometry that

$$\sigma(C(\epsilon)) = \frac{\mathsf{Area}(C(\epsilon))}{\mathsf{Area}(S^{n-1})} = \frac{\mathsf{Vol}(T)}{\mathsf{Vol}(B_2^n)} \leq \tag{3}$$

$$\frac{\mathsf{Vol}(\sqrt{1-\epsilon^2}B_2^n)}{\mathsf{Vol}(B_2^n)} \leq (\sqrt{1-\epsilon^2})^n \leq \mathsf{e}^{-n\epsilon^2/2}. \tag{4}$$

Therefore, the area of the cap is small. But this means also that the area of two caps must be small. Thus the area is concentrated around the (any) equator. Let E_ϵ be the ϵ-neighborhood of the equator. Then by the Lemma,

$$\sigma(E_\epsilon) \geq 1 - 2\mathsf{e}^{-n\epsilon^2/2}. \tag{5}$$

Thus most of the area of the sphere is in the neighborhood around the equator of width $O(\frac{1}{\sqrt{n}})$.

3 Isoperimetric Inequalities

We begin this section with the classical isoperimetric inequality which dates back to antiquity.

Theorem 2 (Classical Isoperimetric Inequality)**.** *Among all sets in $\mathbb{R}^n$ of a given volume, the Euclidean balls minimize the surface area.*

We can strengthen this isomperimetric inequality by defining first the neighborhood of a set. For a set A and $\epsilon > 0$, set $A_\epsilon = \{x : \mathrm{d}(x, A) \leq \epsilon\}$ where d denotes the Euclidean or geodesic metric. Then we have a stronger isoperimetric inequality:

Theorem 3 (Stronger Isoperimetric Inequality)**.** *For a fixed $\epsilon > 0$, among all sets A of $\mathbb{R}^n$ of a given volume, the Euclidean balls minimize the volume of A_ϵ.*

Note that by taking $\epsilon \to 0$ in the above theorem we recover the Classical Isomperimetric Inequality. This is due to the fact that

$$\mathsf{Area}(A) = \lim_{\epsilon \to 0} \frac{\mathsf{Vol}(A_\epsilon) - \mathsf{Vol}(A)}{\epsilon} \tag{6}$$

Finally, we state an even stronger isoperimetric inequality due to Levy.

Theorem 4 (Levy's Lemma)**.** *Among all subsets A of S^{n-1} of a given area, the spherical caps minimize the area of A_ϵ.*

Note that in Levy's Lemma, the metric used to define A_ϵ may be taken to be the Euclidean or geodesic metric.

We now combine the above results. Let H denote the hemisphere of S^{n-1}. Let A be a subset of S^{n-1} with $\sigma(A) = \frac{1}{2}$. Then by the above theorems we have

$$\sigma(A_\epsilon) \geq \sigma(H_\epsilon) = 1 - \sigma(C(\epsilon)) \geq 1 - \mathsf{e}^{-n\epsilon^2/2}. \tag{7}$$

Thus we have the following theorem.

Theorem 5 (Concentration of Measure)**.** *Let $A \subset S^{n-1}$ and $0 < \epsilon < 1$. If $\sigma(A) \geq \frac{1}{2}$ then $\sigma(A_\epsilon) \geq 1 - \mathsf{e}^{-n\epsilon^2/2}$.*

In words, this says that the area on the sphere is concentrated around every set of measure $\frac{1}{2}$. Also, it has been shown that $\frac{1}{2}$ can be replaced with any constant $c > 0$. See [2].

References

[1] K. Ball. *An elementary introduction to modern convex geometry.* Cambridge U. Press, New York, 1997.

[2] M. Ledoux. *Concentration of Measure Phenomenon.* Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2001.

[3] J. Matousek. *Lectures on Discrete Geometry.* Springer, Berlin-Heidelberg New York, 2002.