Non-Asymptotic Theory of Random Matrices

Lecture 3: Concentration of Measure (cont'd)

Lecturer: Roman Vershynin Scribe: Brendan Farrell

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1 Introduction

In this lecture we establish a functional form of the concentration of measure. Recall that a function $f : X \to Y$ is called 1-Lipschitz if for all x_1 and x_2 in X, $d_Y(f(x_1), f(x_2)) \leq d_X(x, y)$. We will see that every Lipschitz function on S^{n-1} is nearly constant on most of S^{n-1}. Our goal is to look at expectation on S^{n-1},

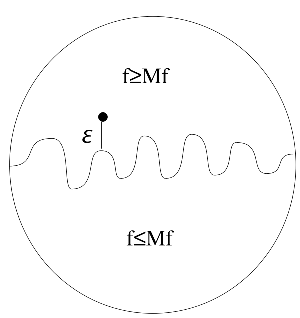
$$\mathbb{E}f = \int_{S^{n-1}} f d\sigma.$$

But first we will work with the median $\mathbb{M}f$, which is defined to be a number satisfying

$$\sigma(f \leq \mathbb{M}f) \geq 1/2 \text{ and } \sigma(f \geq \mathbb{M}f) \geq 1/2.$$

Exercise 1. *Show that $\mathbb{M}f$ always exists for all f.*

(Note: $\mathbb{M}f$ may not be unique, for example when f has a jump.) If f is a 1-Lipschitz function, and x is point at most ϵ-distant from the set $(f \leq \mathbb{M}f)$, then $f(x) \leq \mathbb{M}f + \epsilon$. Therefore, by the Concentration of Measure Theorem, most x on S^{n-1} satisfy $f(x) \leq \mathbb{M}f + \epsilon$. Also, a lower bound is



given by $\sigma(f \le \mathbb{M}f + \epsilon) \ge 1 - e^{-n\epsilon^2/2}$. By applying this same argument to $f \ge \mathbb{M}f$, we also have $\sigma(f(x) \ge \mathbb{M}f - \epsilon) \ge 1 - e^{-n\epsilon^2/2}$, and we conclude that $\sigma(|f - \mathbb{M}f| \le \epsilon) \ge 1 - 2e^{-n\epsilon^2/2}$.

2 Concentration of Measure on S^{n-1} in Functional Form

Theorem 2. *Let $f : S^{n-1} \to \mathbb{R}$ be a $1-$Lipschitz function. Then for all $\epsilon \ge 0$,*

$$\sigma(|f - \mathbb{M}f| \le \epsilon) \ge 1 - 2e^{-n\epsilon^2/2}. \quad (1)$$

We interpret this theorem as follows: "A Lipschitz function is almost a constant on most of the sphere."

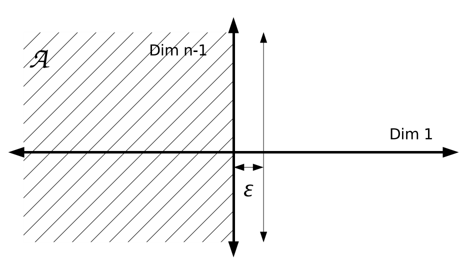
Exercise 3. *Generalize the previous Theorem for L-Lipschitz function. Namely, show that if f is L-Lipschitz, then $\sigma(|f - \mathbb{M}f| \le \epsilon) \ge 1 - 2e^{-n\epsilon^2/2L^2}$).*

Note that in the earlier discussion of the concentration of measure on S^{n-1}, we used very little geometry.

We will establish the concentration of measure for the Gauss space. Gauss space is $\mathbb{R}^n$, where the matric is the standard Euclidean metric, $\|x - y\|_2$, and the measure is the canonical Gaussian measure γ with density $\frac{1}{(2\pi)^{n/2}} e^{-\|x\|_2^2/2} dx$.

Theorem 4 (Isoperimetric Inequality, C. Borell 1975, [1])**.** *For all $\epsilon > 0$ and all sets A satisfying $\gamma(A) = 1/2$, $\gamma(A_\epsilon)$ is minimized when A is a half-space.*

A half-space A in $\mathbb{R}^n$ is a full subspace $\mathbb{R}^{n-1}$ and all $x \le 0$ in the n^{th} dimension, as in the figure.



Exercise 5. *Verify the following:*

$$\begin{aligned}\gamma(A_\epsilon) &= \int_{-\infty}^{\epsilon} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\ &= 1 - \int_{\epsilon}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\ &\geq 1 - e^{-\epsilon^2/2}\end{aligned}$$

Exercise 6. *Compare* $\gamma(A_\epsilon) - \gamma(A)$ *to* $1 - \int_\epsilon^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$*, where* A *is the sphere of measure* $1/2$*.*

Theorem 7 (Concentration of Measure in Gauss Space (functional form))**.** *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a* 1*-Lipschitz function. Then for all* $\epsilon > 0$*,*

$$\gamma(|f - \mathbb{M}f| \leq \epsilon) \geq 1 - 2e^{-2\epsilon^2/2} \tag{2}$$

We will use part of the following lemma to relate $\mathbb{E}f$ and $\mathbb{M}f$.

Lemma 8 (Tails/Integrability/Moments)**.** *Let* X *be a random variable. The following are equivalent:*

1. $\mathbb{P}(X > t) < C_1 e^{-c_1 t^2}$
2. $\mathbb{E}e^{c_2 X^2} \leq C_2$
3. $(\mathbb{E}X^p)^{1/p} \leq C_3\sqrt{p}$ *for* $1 \leq p < \infty$

Where (C_1, c_1)*,* (C_2, c_2) *and* C_3 *are positive constants depending only on each other.*

Proof:
$(1 \Rightarrow 3)$ $\mathbb{E}X = \int_0^\infty \mathbb{P}(X > t)dt$, and $\mathbb{E}X^p = \int_0^\infty \mathbb{P}(X^p > t)dt$. Make the following change of variables: $t = s^p$ and $dt = ps^{p-1}ds$. Then, using this change of variables and Property (1) above,

$$\begin{aligned}\mathbb{E}X^p &= \int_0^\infty \mathbb{P}(X^p > t)dt \\ &\leq p\int_0^\infty C_1 e^{-c_1 s^2} s^{p-1} ds \qquad \text{(note: the integrand is the } \Gamma \text{ function)} \\ &\leq p(c_1 p)^{p/2}\end{aligned}$$

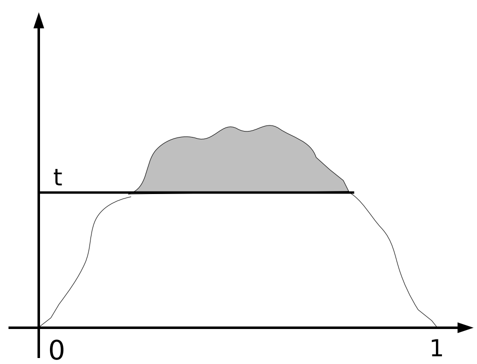
$(3 \Rightarrow 2)$ Use the Taylor expansion: $e^x = \sum_{p=0}^{\infty} \frac{x^p}{p!}$.

$$\begin{aligned}
\mathbb{E}e^{c_2 X^2} &= \sum_{p=0}^{\infty} \frac{c_2^p \mathbb{E}[X^{2p}]}{p!} \\
&\leq \sum_{p=0}^{\infty} \frac{c_2^p (C_3\sqrt{2p})^{2p}}{p!} \\
&\leq 2 \text{ by choosing } c_2 \text{ small.}
\end{aligned}$$

$(2 \Rightarrow 1)$ We will use Markov's inequality: $\mathbb{P}(X > t) \leq \frac{\mathbb{E}X}{t}$.

$$\begin{aligned}
\mathbb{P}(X > t) &= \mathbb{P}(e^{c_2 X^2} > e^{c_2 t^2}) \\
&\leq \frac{\mathbb{E}e^{c_2 X^2}}{e^{c_2 t^2}} \\
&\leq C_2 e^{-c_2 t^2}
\end{aligned}$$

We will use the first property to replace $\mathbb{M}f$ by $\mathbb{E}f$ in the Concentration of Measure.



Lemma 9. *Let f be a 1-Lipschitz function defined on Gauss space. Then there exists a constant C such that $|\mathbb{E}f - \mathbb{M}f| \leq C$.*

Proof: Define the random variable $X = |f - \mathbb{M}f|$. By the concentration of measure, X satisfies property (1) of Lemma 8. Therefore,

$$\begin{aligned} |\mathbb{E}f - \mathbb{M}f| &= |\mathbb{E}(f - \mathbb{M}f)| \\ &\leq \mathbb{E}|f - \mathbb{M}f| \text{ Jensen's Inequality} \\ &\leq C_3 \end{aligned}$$

Lemma 10. *For the sphere, $|\mathbb{E}f - \mathbb{M}f| \leq c/\sqrt{n}$.*

Exercise 11. *Prove the equivalent form ofLemma 9 for the sphere. Start by defining the random variable $X = \sqrt{n}|f - \mathbb{M}f|$.*

The general intuition for concentration of measure theorems is that they will hold on spaces with sufficient symmetries, such as S_n, the symmetric group. See [2].

References

[1] C. Borell. The Brunn-Minkowski inequality in Gauss space. *Invent. Math.*, 30(2):207–216, 1975.

[2] V. D. Milman and G. Schechtman. *Asymptotic theory of finite-dimensional normed spaces*, volume 1200 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.