Non-Asymptotic Theory of Random Matrices
Lecture 6: Norm of a Random Matrix

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1 Introduction

Let A be an $m \times n$ random subgaussian matrix. That is, its entries x_{ij} are i.i.d. mean zero subgaussian random variables, or equivalently, they are independent copies of a mean zero subgaussian random variable X.

We know that the operator norm$\|A\| = s_1(A) = \max_{x \in S^{n-1}} \|Ax\|_2$, and we want to bound this with high probability.

- LOWER ESTIMATES (trivial): $\|A\| \geq \max(\|columns\|_2, \|rows\|_2)$

  In particular, if A is a Bernoulli matrix, then $\|A\| \geq \max(\sqrt{n}, \sqrt{m})$.

- MATCHING UPPER ESTIMATES: norm of a random matrix $\approx$ norms of rows or columns

  - *Asymptotic*: Consider the mode when $n \to \infty$, $\frac{m}{n} \to y$. Let X be a random variable with $\mathbb{E}X^2 = 1$.(*X need not be subgaussian in this case.*) The best asymptotic result is that

    if $\mathbb{E}X^4 < \infty$, then $\frac{\|A\|}{\sqrt{n}} \to 1 + \sqrt{y}$ *a.s.*;
    if $\mathbb{E}X^4 = \infty$, then $\frac{\|A\|}{\sqrt{n}} \to \infty$ *a.s.*.

    For more details, see [5] [2] [1] [8].

  - *Non-asymptotic*: We have the following main theorem in this lecture.

Theorem 1 (Upper Bound)**.** *Let A be an $m \times n$ subgaussian random matrix. Then $\|A\| \leq C(\sqrt{n} + \sqrt{m})$ with probability $1 - \exp\{-c(m+n)\}$.*

Before we get to its proof, let us first pick up some useful tools.

2 Discretization of the Sphere

In the investigation of $\|A\|$, the difficulty is that the maximum " $\max\limits_{x \in S^{n-1}}$ " is taken over infinitely many random variables. This is also called a random process. There are various tools to bound such random processes, from the ϵ-net method (simplest) to the majorizing measure method (hardest). We will work through the ϵ-net method in this lecture. For the majorizing measure method, see [3].

Idea: In order to reduce to a finite situation, we want first to discretize S^{n-1}, or in other words, to replace the sphere with a finite subset. This is possible because the sphere S^{n-1} is compact, and every compact set has a finite ϵ-net.

Definition 2. *An* ***ϵ-net*** *of a subset K of a Banach space is a set $\mathcal{N}$ such that $\forall z \in K, \exists x \in \mathcal{N} : \|z - x\| \leq \epsilon$.*

Note: If $\mathcal{N}$ is an ϵ-net, then every point of K is within distance ϵ from $\mathcal{N}$, and K is covered by balls of radius ϵ centered at $x \in \mathcal{N}$ (these balls are simply translates of $D := \epsilon\ B_X$).

Definition 3. *The minimal cardinality of an ϵ-net of K is called the* ***covering number*** *of K by ϵ-balls,denoted by $N(K, D)$. In other words,*

$$N(K, D) := \textit{minimal number of translates of } D \textit{ to cover } K.$$

Remark: The covering number gives a quantitative notion of compactness.

However, the covering number is hard to compute precisely even in the simplest case where we try to cover the unit disk by smaller disks in $\mathbb{R}^2$. So we may want to make estimates. The good news is that relatively sharp estimates exist.

- LOWER ESTIMATE: $\mathsf{Vol}(K) \leq N(K, D) \cdot \mathsf{Vol}(D) \Rightarrow N(K, D) \geq \frac{\mathsf{Vol}(K)}{\mathsf{Vol}(D)}$.
- MATCHING UPPER ESTIMATE:

Proposition 4 (Covering Number)**.** $N(K, D) \leq \frac{\mathsf{Vol}(K) + \frac{1}{2}D)}{\mathsf{Vol}(\frac{1}{2}D)}$

Proof. (constructive proof — greedy algorithm to find an ϵ-net of K)

We want to locate the centers in order to find the ϵ-net.
Start with an arbitrary point $x_1 \in K$.

Then choose $x_2 \in K$: $\|x_2 - x_1\| > \epsilon$.
Then choose $x_3 \in K$: $\|x_3 - x_k\| > \epsilon, k = 1, 2$.
...
Then choose $x_N \in K$: $\|x_N - x_k\| > \epsilon, k = 1, \dots N-1$.
Stop if no more points are left.

Claim: $\mathcal{N} = \{x_1, \dots, x_N\}$ is an ϵ-net.
Suppose not. Then $\exists z \in K$ such that $\|z - x_k\| > \epsilon \quad \forall k$.
This contradicts the stopping criterion of the algorithm.

Then, if we shrink the radius of these balls to $\epsilon/2$, they will be disjoint (recall that each pair (x_i, x_j) is at least ϵ-apart). Thus, $\epsilon/2$-balls (i.e. $\frac{1}{2}D$) centered at points of $\mathcal{N}$ are disjoint and contained in $K + \frac{1}{2}D$. Therefore,

$$\mathsf{Vol}(K + \frac{1}{2}D) \geq N \cdot \mathsf{Vol}(\frac{1}{2}D),$$

and thus

$$N \leq \frac{\mathsf{Vol}(K + \frac{1}{2}D)}{\mathsf{Vol}(\frac{1}{2}D)}.$$

$\square$

Remark:An upper bound for $vol(K + \frac{1}{2}D)$ was found by Bourgain-Milman [7].

Example 5. *Let K be the unit ball B_X in $\mathbb{R}^n$. We want to cover the unit ball with smaller balls $D = \epsilon B_X$. Note that*

$$K + \frac{1}{2}D = B_X + \frac{1}{2}\epsilon B_X = (1 + \frac{1}{2}\epsilon)B_X.$$

Then, the covering number

$$N(K, D) \leq \frac{(1 + \frac{1}{2}\epsilon)^n}{(\frac{1}{2}\epsilon)^n}$$

Corollary 6. *Let $\epsilon \in (0,1)$. For n-dimensional Banach spaces, the unit ball has an ϵ-net of cardinality $(\frac{3}{\epsilon})^n$.*

Remark: We can follow the same procedure to formulate a statement for the unit sphere.

Proposition 7 (Discretization of $\|A\|$)**.** *Let $\mathcal{N}$ be a $\frac{1}{2}$-net of S^{n-1}. Then $\|A\| \leq 2 \max_{x \in \mathcal{N}} \|Ax\|_2$.(For simplicity, here we let $\epsilon = \frac{1}{2}$)*

Proof. Note that every $z \in S^{n-1}$ can be written as $z = x + u$, where $x \in \mathcal{N}$, $\|u\| \leq \epsilon = \frac{1}{2}$. Then, by the triangle inequality,

$$\|A\| \leq \max_{x \in \mathcal{N}} \|Ax\|_2 + \max_{u \,:\, \|u\|_2 \leq \frac{1}{2}} \|Au\|_2$$

But $\displaystyle\max_{u \,:\, \|u\|_2 \leq \frac{1}{2}} \|Au\|_2 = \frac{1}{2}\|A\|$ by definition. So we have

$$\|A\| \leq \max_{x \in \mathcal{N}} \|Ax\|_2 + \frac{1}{2}\|A\|,$$

and therefore,

$$\|A\| \leq 2 \max_{x \in \mathcal{N}} \|Ax\|_2$$

□

Note: (Exercise) Further, we can discretize the Euclidean norm

$$\|Ax\|_2 \;=\; \max_{y \in S^{m-1}} \langle Ax, y \rangle \;\leq\; 2 \max_{y \in \mathcal{M}} \langle Ax, y \rangle,$$

where $\mathcal{M}$ is the $\frac{1}{2}$-net of S^{m-1}.

Corollary 8. *Let $\mathcal{N}, \mathcal{M}$ be $\frac{1}{2}$-nets of S^{n-1}, S^{m-1}, respectively. Then,*

$$\|A\| \;\leq\; 4 \max_{\substack{x \in \mathcal{N} \\ y \in \mathcal{M}}} \langle Ax, y \rangle.$$

Note: We can find that $|\mathcal{N}| \;\leq\; 6^n$ and $|\mathcal{M}| \;\leq\; 6^n$ by Corollary 6 with $\epsilon = \frac{1}{2}$.

3 Proof of the Main Theorem

Now,we are ready to prove the main theorem that gives an upper estimate of $\|A\|$.

Proof. Let $t > 0$. By Corollary 8, we have

$$\mathbb{P}(\|A\| > \; t) \leq \mathbb{P}(\max_{\substack{x \in \mathcal{N} \\ y \in \mathcal{M}}} \langle Ax, y \rangle \;>\; \frac{t}{4}).$$

Note that if $\displaystyle\max_{\substack{x \in \mathcal{N} \\ y \in \mathcal{M}}} \langle Ax, y \rangle \;>\; \frac{t}{4}$, then $\exists\, x \in \mathcal{N}$ and $y \in \mathcal{M}$ such that

$$\langle Ax, y \rangle > \frac{t}{4}.$$

In other words, the event $(\max_{\substack{x\in\mathcal{N}\\ y\in\mathcal{M}}}\langle Ax,y\rangle > \frac{t}{4}) \subseteq \bigcup_{\substack{x\in\mathcal{N}\\ y\in\mathcal{M}}} (\langle Ax,y\rangle \ > \frac{t}{4})$. Hence,

$$\mathbb{P}(\max_{\substack{x\in\mathcal{N}\\ y\in\mathcal{M}}}\langle Ax,y\rangle \ > \ \frac{t}{4}) \leq \sum_{\substack{x\in\mathcal{N}\\ y\in\mathcal{M}}} \mathbb{P}(\langle Ax,y\rangle \ > \ \frac{t}{4})$$

Now,fix any $x \in \mathcal{N}, y \in \mathcal{M}$, and note that $\langle Ax,y\rangle$ is a random variable.

<u>*Claim*</u>: $\langle Ax,y\rangle$ is a subgaussian random variable.

Let $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, and let X_{ij} denote the ij^{th} entry in the random matrix A. Then, the coordinates of Ax are

$$\sum_{j=1}^{n} X_{ij}x_{ij}.$$

Notice that $x \in S^{n-1}$ so $\sum_{j=1}^{n} x_j = 1$. Also, X_{ij} are independent mean zero subgaussian random variables.
Then, by Theorem 5 from last lecture, the coordinates of Ax are independent subgaussian random variables with mean zero. Then,

$$\langle Ax,y\rangle = \sum_{i=1}^{n}(Ax)_i \, y_i,$$

where $\sum_{i=1}^{n} y_i = 1$. Using the same theorem again, we see that $\langle Ax,y\rangle$ is subgaussian.

Now, knowing that $\langle Ax,y\rangle$ is subgaussian, we have, by definition,

$$\mathbb{P}(\langle Ax,y\rangle \ > \ \frac{t}{4}) \ \leq \ C\exp(-ct^2).$$

Then,

$$\sum_{\substack{x\in\mathcal{N}\\ y\in\mathcal{M}}} \mathbb{P}(\langle Ax,y\rangle \ > \ \frac{t}{4}) \ \leq \ |\mathcal{N}|\cdot|\mathcal{M}|\cdot C\exp(-ct^2) \ \leq \ 6^{m+n}\cdot C\exp(-ct^2)$$

Thus,with $t = C(\sqrt{n}+\sqrt{m})$, we obtain the desired result.

□

Remark:

1. This proof is due to Litvak-Pajar-Rudelson-Tomczak-Vershynin [6].
2. The entries of A must essentially be subgaussian for this result to hold. (Slight fluctuation is allowed)
3. Open problem: (fluctuations) $\mathbb{P}\big(\big|\|A\| - \mathbb{E}\|A\|\big| \ > \ t\big) \ \leq ?$

   - Best result by Meckes [4]
   - If A is gaussian, then $\mathbb{E}\|A\| \leq \sqrt{n} + \sqrt{m}$.

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