MAT 21B-A, Spring 2007, Prof. Opmeer
Thursday June 14, 2007.

Final

NAME (print in CAPITAL LETTERS, last name first):

Solutions

ID#:

Instructions:

- Read each question carefully and answer in the space provided. If the space provided is not enough, please continue on the opposite (blank) side and CLEARLY INDICATE this.
- YOU MUST GIVE CLEAR AND REASONABLY COMPLETE ANSWERS TO RECEIVE FULL CREDIT. Answers like 'yes' or 'no' or '2' will be awarded no credit. Proper notation and (mathematical) readability of your answers play a role in determining credit.
- Calculators, books, notes or similar things are not allowed.
- The numbers between brackets refer to the number of points (out of 100) for that exercise.

1.[30] Compute the following indefinite integrals.
Indicate the substitution that you make (if you make one) and state the trigonometric identity that you use (if you use one).

(a)[6] $\int (x+1)^2 e^x \, dx$

$= (x+1)^2 e^x - 2\int (x+1) e^x \, dx$

$u = (x+1)^2 \quad dv = e^x \, dx$
$du = 2(x+1)\,dx \quad v = e^x$

$= (x+1)^2 e^x - 2\left[(x+1)e^x - \int e^x \, dx\right]$

$u = x+1 \quad dv = e^x dx$
$du = dx \quad v = e^x$

$= \boxed{(x+1)^2 e^x - 2(x+1)e^x + 2e^x + C}$

(b)[6] $\int \frac{x}{x^2-3x+2} \, dx$

$\frac{x}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$

$x = A(x-1) + B(x-2)$

$x=1$: $1 = -B \Rightarrow B = -1$

$x=2$: $2 = A$

$= \int \left(\frac{2}{x-2} + \frac{-1}{x-1} \right) dx$

$= \boxed{2\ln|x-2| - \ln|x-1| + C}$

1.[continued]
(c)[6] $\int \sin^3 x \cos^4 x\, dx$

$= \int \sin^2 x \cos^4 x \sin x\, dx$ $\quad$ $u = \cos x$
$\quad$ $du = -\sin x\, dx$

$= -\int (1-\cos^2 x) \cos^4 x\, (-\sin x) dx$

$= -\int (1-u^2) u^4\, du$

$= -\int (u^4 - u^6)\, du$

$= -\left[\frac{1}{5}u^5 - \frac{1}{7}u^7\right] + C$

$= \boxed{-\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C}$

(d)[6] $\int \frac{\ln\sqrt{x}}{x}\, dx$

$= \frac{1}{2}\int \frac{\ln x}{x}\, dx$ $\quad$ $u = \ln x$
$\quad$ $du = \frac{1}{x} dx$

$= \frac{1}{2}\int u\, du$

$= \frac{1}{2}\frac{u^2}{2} + C$

$= \boxed{\frac{1}{4}(\ln x)^2 + C}$

$\tan^2\theta = \sec^2\theta - 1$

1.[continued]

(e)[6] $\int \frac{1}{(x^2-1)^{3/2}}\,dx$

$x = \sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$

$$= \int \frac{1}{(\sec^2\theta - 1)^{3/2}} \sec\theta\tan\theta\,d\theta$$

$$= \int \frac{1}{\tan^3\theta} \sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \int \frac{1}{\cos\theta}\,\frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= \int \frac{1}{u^2}\,du$$

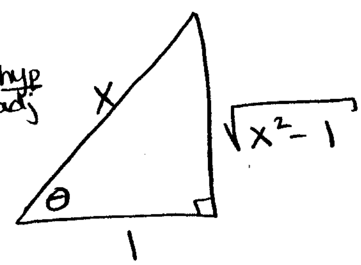
$$= -\frac{1}{u} + C$$

$$= \frac{-1}{\sin\theta} + C$$

$$= \frac{-1}{\frac{\sqrt{x^2-1}}{x}} + C$$

$$= \boxed{\frac{-x}{\sqrt{x^2-1}} + C}$$

$\frac{x}{1} = \sec\theta = \frac{hyp}{adj}$



2.[12] Evaluate the following improper integrals.

(a)[6] $\int_1^\infty e^{-x}\,dx$

$$= \lim_{b\to\infty} \int_1^b e^{-x}\,dx$$

$$= \lim_{b\to\infty} -e^{-x}\Big|_1^b$$

$$= \lim_{b\to\infty} \left(-e^{-b} + e^{-1}\right)$$

$$= \lim_{b\to\infty} \left(-\cancel{\frac{1}{e^b}}^{\to 0} + \frac{1}{e}\right)$$

$$= \boxed{\frac{1}{e}}$$

$$\int e^{-x}\,dx$$

$$= -e^{-x} + C$$

(b)[6] $\int_0^\infty \frac{1}{(1+x)\sqrt{x}}\,dx$

$$= \lim_{a\to 0^+} \int_a^2 \frac{1}{(1+x)\sqrt{x}}\,dx + \lim_{b\to\infty} \int_2^b \frac{1}{(1+x)\sqrt{x}}\,dx$$

$$= \lim_{a\to 0^+} \left[2\arctan\sqrt{x}\,\Big|_a^2\right] + \lim_{b\to\infty} \left[2\arctan\sqrt{x}\,\Big|_2^b\right]$$

$$= \lim_{a\to 0^+} \left[2\tan^{-1}\sqrt{2} - 2\tan^{-1}\sqrt{a}\right]$$

$$+ \lim_{b\to\infty} \left[2\tan^{-1}\sqrt{b} - 2\tan^{-1}\sqrt{2}\right]$$

$$= \left(2\tan^{-1}\sqrt{2} - 0\right) + \left(2\cdot\frac{\pi}{2} - 2\tan^{-1}\sqrt{2}\right)$$

$$= \boxed{\pi}$$

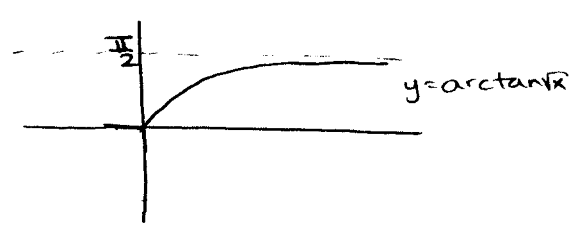
$$\int \frac{1}{(1+x)\sqrt{x}}\,dx$$

$$= \int \frac{1}{(1+(\sqrt{x})^2)\sqrt{x}}\,dx \qquad u = \sqrt{x}, \quad du = \frac{1}{2\sqrt{x}}\,dx$$

$$= 2\int \frac{1}{1+u^2}\,du$$

$$= 2\arctan u + C$$

$$= 2\arctan\sqrt{x} + C$$



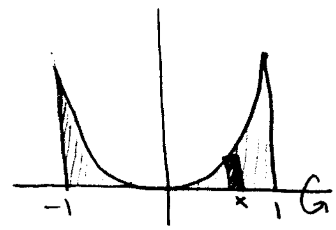
3.[12] Consider the region bounded by the x-axis, the graph $y = 3x^4$ and the lines $x = -1$ and $x = 1$. Set-up, but do not evaluate, integrals which represent the volume of the solid formed by revolving this region about

(a)[3] the x-axis using the disc method.

ACS: solid cylinder

Vol: $\pi r^2 h$

$$\boxed{V = \int_{-1}^{1} \pi (3x^4)^2 \, dx}$$

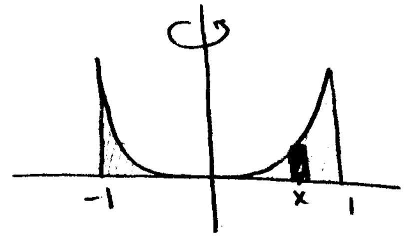


(b)[3] the y-axis using the shell method.

ACS: shell

Vol: $2\pi r \cdot h \cdot$ thickness

$$\boxed{V = \int_{0}^{1} 2\pi \cdot x \, (3x^4) \, dx}$$

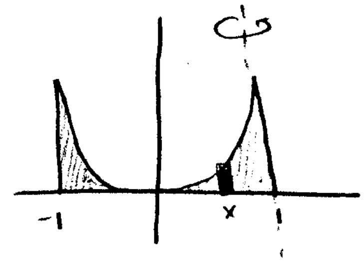


(c)[3] the line $x = 1$ using the shell method.

ACS: shell

Vol: $2\pi r \cdot h \cdot$ thickness

$$\boxed{V = \int_{-1}^{1} 2\pi (1-x)(3x^4) \, dx}$$

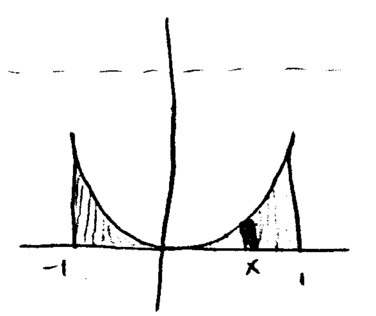


(d)[3] the line $y = 3$ using the disc method.

ACS: hollow cylinder

Vol: $\pi R^2 h - \pi r^2 h$

$$\boxed{V = \int_{-1}^{1} \pi (3)^2 dx - \int_{-1}^{1} \pi (3 - 3x^4)^2 dx}$$



$y = 3$

$x=\sqrt{y-1}$

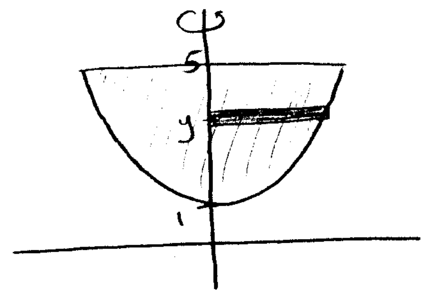
4.[12] Consider the region bounded by the graph $y = x^2 + 1$ and the line $y = 5$. Set-up, but do not evaluate, integrals which represent the volume of the solid formed by revolving this region about

(a)[3] the y-axis using the disc method.

ACS: solid cylinder

Vol: $\pi r^2 h$

$$\boxed{V = \int_1^5 \pi\left(\sqrt{y-1}\right)^2 dy}$$

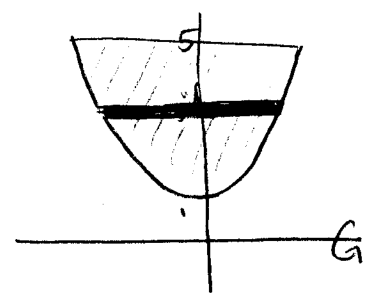


(b)[3] the x-axis using the shell method.

ACS: shell

Vol: $2\pi r \cdot h \cdot$ thickness

$$\boxed{V = \int_1^5 2\pi y\left(2\sqrt{y-1}\right) dy}$$

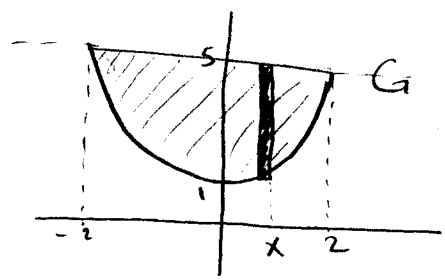


(c)[3] the line $y = 5$ using the disc method.

ACS: solid cylinder

Vol: $\pi r^2 h$

$$\boxed{V = \int_{-2}^{2} \pi\left(5-(x^2+1)\right)^2 dx}$$



$x^2+1=5$

$x^2-4=0 \quad (x-2)(x+2)=0$

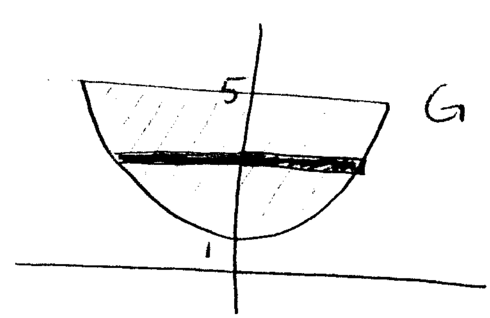
$x=2, \; x=-2$

(d)[3] the line $y = 5$ using the shell method.

ACS: shell

Vol: $2\pi r \cdot h \cdot$ thickness

$$\boxed{V = \int_1^5 2\pi(5-y)\left(2\sqrt{y-1}\right) dy}$$



5.[6] Set-up, but do not evaluate, an integral which gives the length of the following curve.
$x(t) = t^3 - 6t^2, \;\; y(t) = t^3 + 6t^2, \;\; 0 \le t \le 1.$

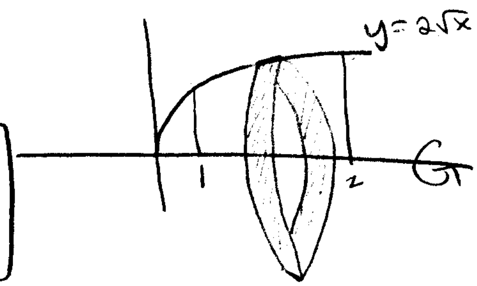
$\frac{dx}{dt} = 3t^2 - 12t, \quad \frac{dy}{dt} = 3t^2 + 12t$

$$\boxed{L = \int_0^1 \sqrt{(3t^2-12t)^2 + (3t^2+12t)^2}\, dt}$$

6.[6] Set-up, but do not evaluate, an integral which gives the area of the surface generated by revolving the curve $y = 2\sqrt{x}, \;\; 1 \le x \le 2$ about the x-axis.

$SA = 2\pi r \cdot \text{arclength}$

$$\boxed{SA = \int_1^2 2\pi (2\sqrt{x}) \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}\, dx}$$

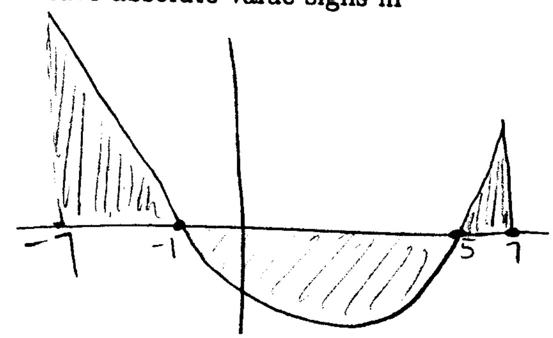


$y = 2\sqrt{x}$

$y' = \frac{1}{\sqrt{x}}$

7.[12] $(x-5)(x+1)$

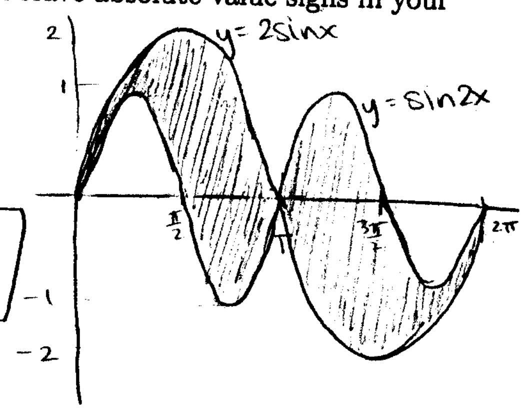
(a)[6] Set-up the area between the x-axis and the graph of the function $x^2 - 4x - 5$ over the interval $[-7, 7]$ in terms of integrals. Do not evaluate. Do not leave absolute value signs in your final answer.



$$\boxed{A = \int_{-7}^{-1} (x^2-4x-5)\,dx + \int_{-1}^{5} (0 - (x^2-4x-5))\,dx + \int_{5}^{7} (x^2-4x-5)\,dx}$$

(b)[6] Set-up the area of the region enclosed by the curves $y = 2\sin x$ and $y = \sin 2x$ with $0 \le x \le 2\pi$ in terms of integrals. Do not evaluate. Do not leave absolute value signs in your final answer.

$$\boxed{A = \int_0^{\pi} (2\sin x - \sin 2x)\,dx + \int_{\pi}^{2\pi} (\sin 2x - 2\sin x)\,dx}$$



$2\sin x = \sin 2x$

$2\sin x = 2\sin x\cos x$

$2\sin x\cos x - 2\sin x = 0$

$2\sin x(\cos x - 1) = 0$

$\sin x = 0, \quad \cos x - 1 = 0$

$\cos x = 1$

$x = 0, \pi, 2\pi \quad x = 0, 2\pi$

8.[10]

(a)[5] Find $f(4)$ if $\int_0^{x^2} f(t)\,dt = x\cos\pi x$.

$$\frac{d}{dx}\left[\int_0^{x^2} f(t)\,dt\right] = \frac{d}{dx}\left[x\cos\pi x\right]$$

$$f(x^2)\cdot 2x = \cos\pi x - \pi x\sin\pi x$$

$$f(x^2) = \frac{\cos\pi x - \pi x \sin\pi x}{2x}$$

$$f(4) = f(2^2) = \frac{\cos 2\pi - \pi\cdot 2\sin 2\pi}{2\cdot 2}$$

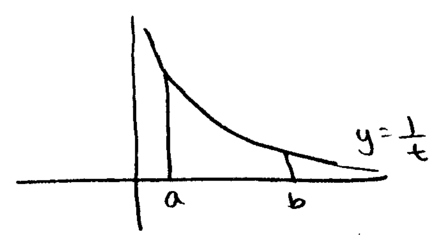
plug in $x=2$

$$= \frac{1-0}{4} = \boxed{\frac{1}{4}}$$

(b)[5] Use the definition of the natural logarithm as an integral to show that for all a and b with $b > a > 0$ the following holds:

$$\frac{1}{b} < \frac{\ln b - \ln a}{b-a} < \frac{1}{a}.$$

$0 < a < b$



def: $\ln x = \int_1^x \frac{1}{t}\,dt$

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$$\frac{f(b)-f(a)}{b-a} = \frac{\ln b - \ln a}{b-a}$$

Recall: Mean Value Thm

$f(x)$ continuous on $[a,b]$, differentiable on (a,b)

$\Rightarrow \frac{f(b)-f(a)}{b-a} = f'(c)$, for some c in (a,b)

$$\frac{\ln b - \ln a}{b-a} = f'(c), \text{ for some } c \text{ in } (a,b)$$

$f'(x) = \frac{1}{x}$

$\frac{1}{b} < \frac{1}{c} < \frac{1}{a}$

$0 < a < c < b$

$$\Rightarrow \quad \frac{1}{b} < \frac{\ln b - \ln a}{b-a} < \frac{1}{a}$$