MAT 21B-A, Spring 2007, Prof. Opmeer
Wednesday Apr. 25, 2007.

MIDTERM EXAM 1

NAME (print in CAPITAL LETTERS, last name first):

Solutions

ID#:

Instructions:

- Read each question carefully and answer in the space provided. If the space provided is not enough, please continue on the opposite (blank) side and CLEARLY INDICATE this.
- YOU MUST GIVE CLEAR AND REASONABLY COMPLETE ANSWERS TO RECEIVE FULL CREDIT. Answers like 'yes' or 'no' or '2' will be awarded no credit. Proper notation and (mathematical) readability of your answers play a role in determining credit.
- Calculators, books, notes or similar things are not allowed.
- The numbers between brackets refer to the number of points (out of 100) for that exercise.

1.[24] Find all antiderivatives of the following functions (i.e. find $\int f(x)\,dx$). Indicate the substitution that you make (if you make one).

(a)[6] $f(x) = 2e^x - 3e^{-2x}$

$$\int (2e^x - 3e^{-2x})\,dx$$

$$= \int 2e^x\,dx - 3\int e^{-2x}\,dx$$

$u = -2x$
$du = -2dx$

$$= \boxed{2e^x + \frac{3}{2}e^{-2x} + C}$$

(b)[6] $f(x) = \frac{4+\sqrt{x}}{x^3}$

$$\int \frac{4+\sqrt{x}}{x^3}\,dx$$

$$= \int (4x^{-3} + x^{-5/2})\,dx$$

$$= \frac{4}{-2}x^{-2} + \frac{-2}{3}x^{-3/2} + C$$

$$= \boxed{-2x^{-2} - \frac{2}{3}x^{-3/2} + C}$$

1.[continued] Find all antiderivatives of the following functions (i.e. find $\int f(x)\,dx$). Indicate the substitution that you make (if you make one).

(c)[6] $f(x) = \frac{1}{\sqrt{x}(1+\sqrt{x})^2}$

$$\int \frac{1}{\sqrt{x}\,(1+\sqrt{x})^2}\,dx$$

$u = 1+\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$$= 2\int \frac{1}{(1+\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}}\,dx$$

$$= 2\int u^{-2}\,du$$

$$= 2\,\frac{u^{-1}}{-1} + C$$

$$= \boxed{-2(1+\sqrt{x})^{-1} + C}$$

(d)[6] $f(x) = e^{x^3}x^2 + \cos x$

$$\int \left(x^2e^{x^3} + \cos x\right)dx$$

$$= \int x^2e^{x^3}\,dx + \int \cos x\,dx$$

$u = x^3$

$du = 3x^2dx$

$$= \frac{1}{3}\int e^{x^3}\cdot 3x^2dx + \int \cos x\,dx$$

$$= \frac{1}{3}\int e^u\,du + \int \cos x\,dx$$

$$= \frac{1}{3}e^u + \sin x + C$$

$$= \boxed{\frac{1}{3}e^{x^3} + \sin x + C}$$

2.[16] Evaluate the following integrals.
Indicate the substitution that you make (if you make one).

(a)[8] $\int_0^4 \left(3x - \frac{x^3}{4}\right)\, dx$

$$\int_0^4 \left(3x - \frac{1}{4}x^3\right) dx$$

$$= \frac{3}{2}x^2 - \frac{1}{16}x^4 \Big|_0^4$$

$$= \frac{3}{2}\cdot 16 - \frac{1}{16}\cdot 4^4 - 0$$

$$= 24 - 16$$

$$= \boxed{8}$$

(b)[8] $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$

$u = \ln x$
$du = \frac{1}{x} dx$

$$\int_2^{16} \frac{1}{2x\sqrt{\ln x}}\, dx$$

$$= \int_2^{16} \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x}\, dx$$

$$= \frac{1}{2}\int_{x=2}^{x=16} u^{-1/2}\, du$$

$$= \frac{1}{2}\cdot \frac{2}{1} u^{1/2} \Big|_{x=2}^{x=16}$$

$$= \sqrt{\ln x}\, \Big|_2^{16}$$

$$= \boxed{\sqrt{\ln 16} - \sqrt{\ln 2}}$$

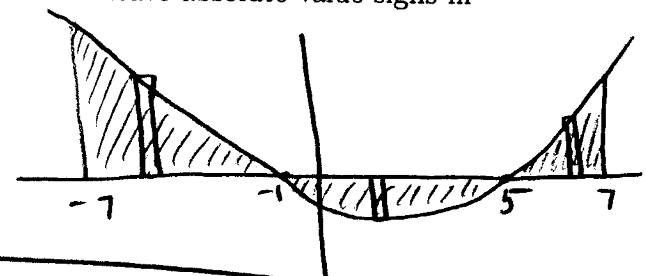
3.[18]

(a)[9] Set-up the area between the x-axis and the graph of the function $x^2 - 4x - 5$ over the interval $[-7, 7]$ in terms of integrals. Do not evaluate. Do not leave absolute value signs in your final answer.

$y = x^2 - 4x - 5$

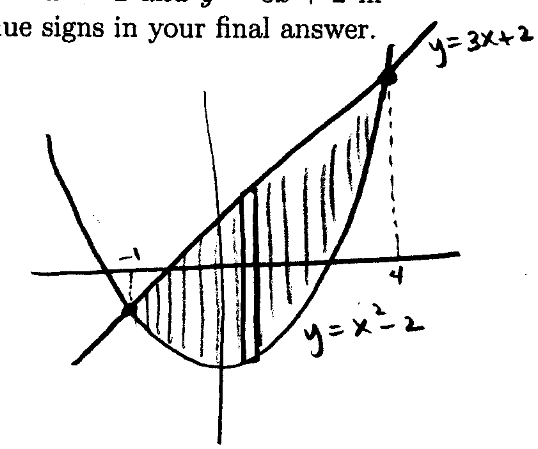
$y = (x-5)(x+1) = 0$

$x = 5,\ x = -1$



$$\boxed{A = \int_{-7}^{-1} (x^2 - 4x - 5)\, dx + \int_{-1}^{5} [0 - (x^2 - 4x - 5)]\, dx + \int_{5}^{7} (x^2 - 4x - 5)\, dx}$$

(b)[9] Set-up the area of the region enclosed by the curves $y = x^2 - 2$ and $y = 3x + 2$ in terms of integrals. Do not evaluate. Do not leave absolute value signs in your final answer.



$x^2 - 2 = 3x + 2$

$x^2 - 3x - 4 = 0$

$(x-4)(x+1) = 0$

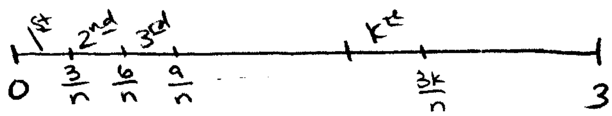
$x = 4,\ x = -1$

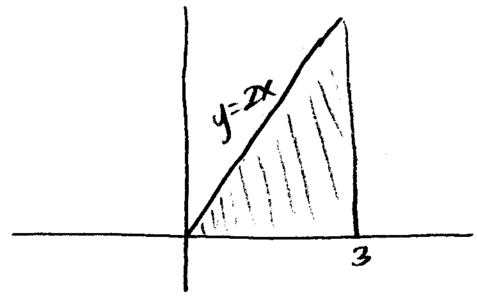
$$\boxed{A = \int_{-1}^{4} [\underbrace{(3x+2)}_{\text{Top}} - \underbrace{(x^2 - 2)}_{\text{Bottom}}]\, dx}$$

4.[18]

(a)[6] Using Sigma notation, express the uppersum and the lowersum of $f(x) = 2x$ obtained by dividing the interval $[0, 3]$ into n equal subintervals. Do not evaluate the sums.

$$\Delta x = \frac{3-0}{n} = \frac{3}{n}$$





$$U = \sum_{k=1}^{n} f\left(\frac{3k}{n}\right)\cdot\frac{3}{n}$$

$$\Rightarrow \boxed{U = \sum_{k=1}^{n} 2\cdot\frac{3k}{n}\cdot\frac{3}{n}} \quad \text{or} \quad \boxed{U = \sum_{k=1}^{n} \frac{18k}{n^2}}$$

$$L = \sum_{k=0}^{n-1} f\left(\frac{3k}{n}\right)\cdot\frac{3}{n}$$

$$\Rightarrow \boxed{L = \sum_{k=0}^{n-1} 2\cdot\frac{3k}{n}\,\frac{3}{n}} \quad \text{or} \quad \boxed{L = \sum_{k=0}^{n-1} \frac{18k}{n^2}}$$

4.[continued]
(b)[6] Suppose that the uppersum and lowersum for a certain function f obtained by dividing a certain interval $[a, b]$ into n equal subintervals are, respectively,

$$U = \sum_{k=1}^{n} \frac{k}{n^2} + \frac{k^2}{n^3}, \qquad L = \sum_{k=0}^{n-1} \frac{k}{n^2} + \frac{k^2}{n^3}.$$

Show that $U - L$ converges to 0 as $n \to \infty$.

$$\lim_{n\to\infty} \left[\sum_{k=1}^{n} \left(\frac{k}{n^2} + \frac{k^2}{n^3} \right) - \sum_{k=0}^{n-1} \left(\frac{k}{n^2} + \frac{k^2}{n^3} \right) \right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n^2} \sum_{k=1}^{n} k + \frac{1}{n^3} \sum_{k=1}^{n} k^2 - \frac{1}{n^2} \sum_{k=0}^{n-1} k - \frac{1}{n^3} \sum_{k=0}^{n-1} k^2 \right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n^2} \cdot \frac{n(n+1)}{2} + \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{1}{n^2} \frac{(n-1)(n)}{2} - \frac{1}{n^3} \frac{(n-1)(n)(2n-1)}{6} \right]$$

$$= \frac{1}{2} \cdot 1 + \frac{1}{6} \cdot 2 - \frac{1}{2} \cdot 1 - \frac{1}{6} \cdot 2$$

$$= 0$$

(c)[6] Does the result in part (b) show that the function f is integrable over $[a, b]$? Be sure to justify your answer.

Yes, since $\lim_{n\to\infty} (U-L) = 0$, then

$$\lim_{n\to\infty} U = \lim_{n\to\infty} L$$

Also, $\lim_{n\to\infty} L \le \int_a^b f(x)\,dx \le \lim_{n\to\infty} U$

these two are equal,

So f is integrable over $[a,b]$.

5.[8] Show that the value of $\int_0^1 \sin x^2\,dx$ cannot possibly be 2.

$$\int_0^1 \sin x^2\,dx \le \int_0^1 1\,dx$$
$$= x\Big|_0^1$$
$$= 1$$
$$\int_0^1 \sin x^2\,dx \le 1 < 2.$$

6.[8] Suppose that f is differentiable on $[a, b]$. Is it true that f is the derivative of some function on $[a, b]$?

Yes!

Since f is differentiable on $[a,b]$, then f is continuous on $[a,b]$. Thus integrable – ie $\int f(x)dx = g(x)$.

$\Rightarrow f(x) = g'(x)$.

7.[8] Calculate $\frac{d}{dx}\int_1^{1+x^2} \frac{\cos t}{1+\ln t}\,dt$. Hint: use the fundamental theorem of calculus and the chain rule, do not find an explicit antiderivative.

$$\frac{d}{dx}\int_1^{1+x^2} \frac{\cos t}{1+\ln t}\,dt$$

$$= \boxed{\frac{\cos(1+x^2)}{1+\ln(1+x^2)} \cdot 2x}$$