MAT 21B-A, Spring 2007, Prof. Opmeer
Wednesday May 9, 2007.

MIDTERM EXAM 2

NAME (print in CAPITAL LETTERS, last name first):

Solutions

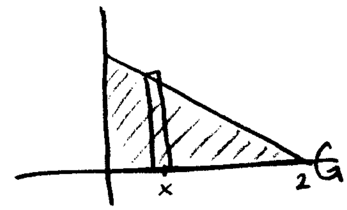
ID#:

Instructions:

- Read each question carefully and answer in the space provided. If the space provided is not enough, please continue on the opposite (blank) side and CLEARLY INDICATE this.
- YOU MUST GIVE CLEAR AND REASONABLY COMPLETE ANSWERS TO RECEIVE FULL CREDIT. Answers like 'yes' or 'no' or '2' will be awarded no credit. Proper notation and (mathematical) readability of your answers play a role in determining credit.
- Calculators, books, notes or similar things are not allowed.
- The numbers between brackets refer to the number of points (out of 100) for that exercise.

1.[24]

(a)[12] The region bounded by the line $x+2y=2$, the x-axis and the y axis is revolved about the x-axis to form a solid. Set-up and evaluate an integral which represents the volume of this solid. $x+2y=2 \Rightarrow y=-\frac{1}{2}x+1$



ACS: solid cylinder

Vol: $\pi r^2 h$

$$V=\int_0^2 \pi\left(-\tfrac{1}{2}x+1\right)^2 dx$$

$u=-\frac{1}{2}x+1$

$du=-\frac{1}{2}dx$

$$=-2\int_0^2 \pi\left(-\tfrac{1}{2}x+1\right)^2\left(-\tfrac{1}{2}dx\right)$$

$-\frac{1}{2}x+1=0$

$\frac{1}{2}x=1$

$\boxed{x=2}$

$$=-2\int_{x=0}^{x=2}\pi u^2\,du$$

$$=-2\pi\left.\frac{u^3}{3}\right|_{x=0}^{x=2}$$

$$=-\frac{2\pi}{3}\left.\left(-\tfrac{1}{2}x+1\right)^3\right|_0^2$$

$$=-\frac{2\pi}{3}\left(0^3-1^3\right)=\boxed{\frac{2\pi}{3}}$$

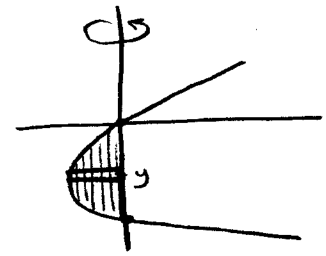
(b)[12] The region bounded by the curve $x=2y+y^2$ and the y-axis is revolved about the y-axis to form a solid. Set-up and evaluate an integral which represents the volume of this solid.

$x=y^2+2y$

$x=y(y+2)$

ACS: solid cylinder

Vol: $\pi r^2 h$



$$V=\int_{-2}^{0}\pi\left(0-(y^2+2y)\right)^2 dy$$

$$=\pi\int_{-2}^{0}\left(y^4+4y^3+4y^2\right)dy$$

$$=\pi\left[\frac{y^5}{5}+y^4+\frac{4}{3}y^3\right]_{-2}^{0}$$

$$=\pi\left[0-\left(\frac{-32}{5}+16-\frac{32}{3}\right)\right]$$

$\frac{32}{5}=6+\frac{2}{5}$

$\frac{32}{3}=10+\frac{2}{3}$

$\Rightarrow -\frac{32}{5}-\frac{32}{3}=-6-10-\underbrace{\frac{2}{5}-\frac{2}{3}}$

$-6-10+\frac{-6-10}{15}$

$$=\pi\left(-\left(16-6-10-\frac{16}{15}\right)\right)$$

$$=\boxed{\frac{16}{15}\pi}$$

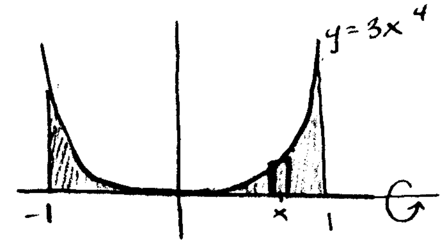
2.[24] Consider the region bounded by the x-axis, the graph $y = 3x^4$ and the lines $x = -1$ and $x = 1$. Set-up, but do not evaluate, integrals which represent the volume of the solid formed by revolving this region about

(a)[6] the x-axis using the disc method.

ACS: solid cylinder

Vol: $\pi r^2 h$

$$\boxed{V = \int_{-1}^{1} \pi (3x^4)^2 \, dx}$$

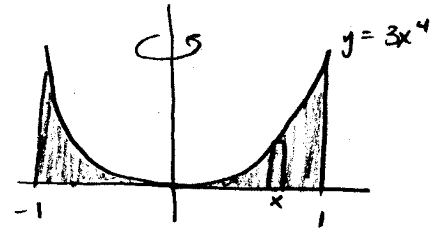


(b)[6] the y-axis using the shell method.

ACS: shell

Vol: $2\pi r \cdot h \cdot$ thickness

$$\boxed{V = \int_{0}^{1} 2\pi x \cdot 3x^4 \cdot dx}$$

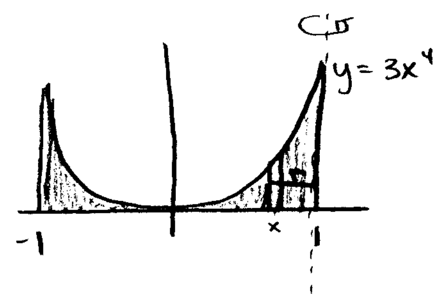


(c)[6] the line $x = 1$ using the shell-method.

ACS: shell

Vol: $2\pi r \cdot h \cdot$ thickness

$$\boxed{V = \int_{-1}^{1} 2\pi (1-x)(3x^4) \, dx}$$

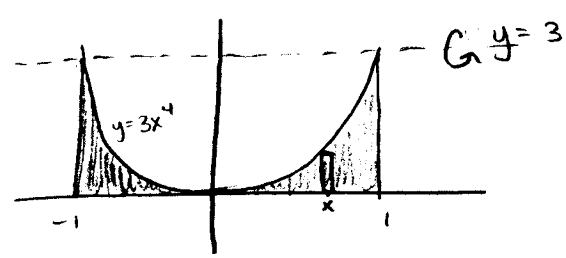


(d)[6] the line $y = 3$ using the disc method.

ACS: hollow cylinder

Vol: $\pi R^2 h - \pi r^2 h$

$$\boxed{V = \int_{-1}^{1} \pi (3)^2 dx - \int_{-1}^{1} \pi (3 - 3x^4)^2 dx}$$



3.[24] Set-up and evaluate integrals which give the lengths of the following curves.

(a)[12] $x(t) = e^t \cos t,\ \ y(t) = e^t \sin t,\ \ 0 \le t \le \pi.$

$x'(t) = e^t\cos t - e^t \sin t \rightarrow (x'(t))^2 = (e^t\cos t - e^t\sin t)^2$

$= e^{2t}\cos^2 t - 2e^{2t}\cos t\sin t + e^{2t}\sin^2 t$

$y'(t) = e^t \sin t + e^t \cos t \rightarrow (y'(t))^2 = (e^t\sin t + e^t\cos t)^2$

$= e^{2t}\sin^2 t + 2e^{2t}\sin t\cos t + e^{2t}\cos^2 t$

$$L = \int_0^\pi \sqrt{\underbrace{e^{2t}\cos^2 t - \cancel{2e^{2t}\cos t\sin t} + e^{2t}\sin^2 t}_{(x'(t))^2} \underbrace{+ e^{2t}\sin^2 t + \cancel{2e^{2t}\cos t\sin t} + e^{2t}\cos^2 t}_{(y'(t))^2}}\, dt$$

$$= \int_0^\pi \sqrt{2e^{2t}(\underbrace{\cos^2 t + \sin^2 t}_{=1})}\, dt$$

$$= \int_0^\pi \sqrt{2}\, e^t\, dt$$

$$= \sqrt{2}\, e^t \Big|_0^\pi = \boxed{\sqrt{2}\,(e^\pi - 1)}$$

(b)[12] $y = \frac{x^3}{6} + \frac{1}{2x},\ \ 2 \le x \le 3.$ Hint: find the perfect square.

$y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$

$y' = \frac{1}{2}x^2 - \frac{1}{2}x^{-2} \longrightarrow (y')^2 = \left(\frac{1}{2}x^2 - \frac{1}{2}x^{-2}\right)^2$

$= \frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4}x^{-4}$

$$L = \int_2^3 \sqrt{1 + \frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4x^4}}\, dx$$

$$= \int_2^3 \sqrt{\frac{1}{4}x^4 + \frac{1}{4x^4} + \frac{1}{2}}\, dx$$

$$= \int_2^3 \sqrt{\frac{x^8 + 1 + 2x^4}{4x^4}}\, dx$$

$$= \int_2^3 \sqrt{\frac{(x^4+1)^2}{(2x^2)^2}}\, dx$$

$$= \int_2^3 \frac{x^4+1}{2x^2}\, dx$$

$$= \frac{1}{2}\int_2^3 (x^2 + x^{-2})\, dx$$

$$= \frac{1}{2}\left[\frac{1}{3}x^3 - \frac{1}{x}\right]_2^3$$

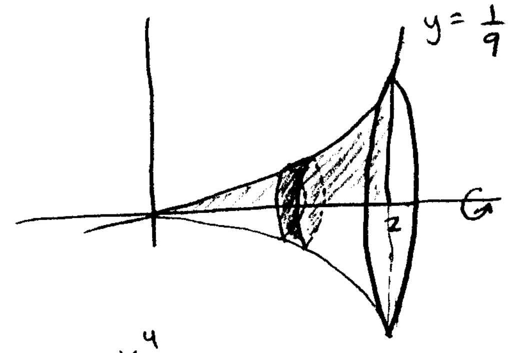
$$= \frac{1}{2}\left[\left(9 - \frac{1}{3}\right) - \left(\frac{8}{3} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\Big[9 \underbrace{- \frac{1}{3} - \frac{8}{3}}_{-\frac{9}{3} = -3} + \frac{1}{2}\Big]$$

$$= \frac{1}{2}\Big[\underbrace{9 - 3}_{6} + \frac{1}{2}\Big]$$

$$= \frac{1}{2}\cdot\frac{13}{2} = \boxed{\frac{13}{4}}$$

4.[12] Set-up and evaluate an integral which gives the area of the surface generated by revolving the curve $y = \frac{x^3}{9}$, $0 \le x \le 2$ about the x-axis.



SA = $2\pi r \cdot$ arclength

$$SA = \int_0^2 2\pi \left(\frac{1}{9}x^3\right)\sqrt{1+\left(\frac{1}{3}x^2\right)^2}\,dx$$

$$= \frac{2\pi}{9}\int_0^2 x^3\sqrt{1+\frac{x^4}{9}}\,dx$$

$u = 1 + \frac{x^4}{9}$

$du = \frac{4}{9}x^3\,dx$

$$= \frac{2\pi}{9}\cdot\frac{9}{4}\int_0^2 \sqrt{1+\frac{x^4}{9}}\cdot\frac{4}{9}x^3\,dx$$

$$= \frac{\pi}{2}\int_{x=0}^{x=2} u^{1/2}\,du$$

$$= \frac{\pi}{2}\cdot\frac{2}{3}\,u^{3/2}\Big|_{x=0}^{x=2}$$

$$= \frac{\pi}{3}\left(1+\frac{x^4}{9}\right)^{3/2}\Big|_0^2$$

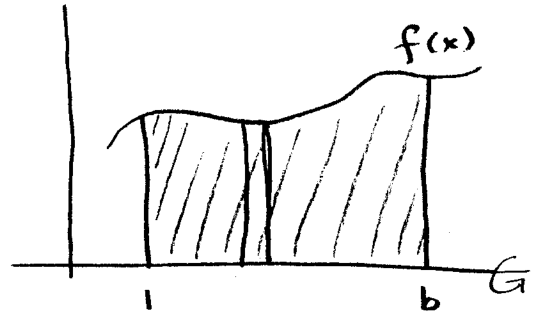
$$= \frac{\pi}{3}\left[\left(1+\frac{16}{9}\right)^{3/2} - (1)^{3/2}\right]$$

$$= \frac{\pi}{3}\left[\left(\frac{25}{9}\right)^{3/2} - 1\right]$$

$$= \frac{\pi}{3}\left[\frac{125}{27} - \frac{27}{27}\right]$$

$$= \boxed{\frac{98\pi}{81}}$$

5.[16] A solid is generated by revolving about the x-axis the region bounded by the graph of the positive continuous function $y = f(x)$, the x-axis, the line $x = 1$ and the line $x = b$ with $b > 1$. Its volume, for all b, is $b^2 - b$. Find $f(x)$.



ACS: solid cylinder

Vol: $\pi r^2 h$

$$V = \int_1^b \pi (f(x))^2 \, dx = b^2 - b$$

looks like $\int \pi (f(x))^2 dx = x^2 - x + C$

$\overset{d/dx}{\Rightarrow} \pi (f(x))^2 = 2x - 1$

$\Rightarrow (f(x))^2 = \dfrac{2x-1}{\pi}$

$\Rightarrow \boxed{f(x) = \sqrt{\dfrac{2x-1}{\pi}}}$

Check: $f(x) = \sqrt{\dfrac{2x-1}{\pi}}$

$$V = \int_1^b \pi \left(\sqrt{\frac{2x-1}{\pi}} \right)^2 dx$$

$$= \int_1^b \pi \left(\frac{2x-1}{\pi} \right) dx$$

$$= \int_1^b (2x-1) \, dx$$

$$= x^2 - x \Big|_1^b$$

$$= b^2 - b - 0$$

$$= b^2 - b \checkmark$$