FINAL EXAM. Solutions

(1)

A sequence of tosses that ends with the first appearance of "HT" has the form

$$\underbrace{TT\ldots TH}_{N_1}\underbrace{H\ldots HT}_{N_2}$$

where N_1 is the number of tosses until the first appearance of H and N_2 is the number of further tosses ~~of~~ until the first app. of T.

The total number of tosses is therefore

$$N := N_1 + N_2 .$$

By Problem 5 of HW6, $\mathbb{E}N_1 = 2$ and $\mathbb{E}N_2 = 2$.
Hence, by the linearity of expectation,

$$\mathbb{E}N = 4 .$$

(2)

Since $Var(X) = \mathbb{E}(X-\mu)^2$ where $\mu = \mathbb{E}X$,
the inequality $Var(X) \geq \min_a \mathbb{E}(X-a)^2$ is trivial.
To prove the converse inequality, i.e.

$$\mathbb{E}(X-a)^2 \geq \mathbb{E}(X-\mu)^2 \quad \text{for all } a,$$

note that

$$\mathbb{E}(X-a)^2 = \mathbb{E}(X-\mu+\mu-a)^2 = \mathbb{E}(X-\mu)^2 + 2(\mu-a)\underbrace{\mathbb{E}(X-\mu)}_{=0} + (\mu-a)^2$$

$$\geq \mathbb{E}(X-\mu)^2 .$$

Q.E.D.

3

(a) $Z_n \to 1$ in probability iff

$$\forall \varepsilon > 0, \quad \mathbb{P}(|Z_n - 1| > \varepsilon) \to 0 \quad \text{as} \quad n \to \infty$$

Since $Z_n = \max\limits_{k \le n} X_n \le 1$, this is equivalent to:

$$\forall \varepsilon > 0, \quad \mathbb{P}(Z_n < 1 - \varepsilon) \to 0 \quad \text{as} \quad n \to \infty.$$

Now,
$$\mathbb{P}(Z_n < 1-\varepsilon) = \mathbb{P}(X_k < 1-\varepsilon \text{ for } k = 1, 2, \ldots n)$$
$$= \prod_{k=1}^{n} \mathbb{P}(X_k < 1-\varepsilon) \quad \text{by independence}$$
$$= (1-\varepsilon)^n \to 0 \quad \text{as} \quad n \to \infty$$

Q.E.D

(b) By definition, Z_n is monotone ~~incr~~ (non-decreasing).
By Corollary 17.6 in the lecture notes,

$$Z_n \to 1 \text{ in probability} \iff Z_n \to 1 \text{ almost surely.}$$

Therefore, the conclusion follows from part (a).

(a) and (b) together:

We first prove:

CLAIM Let X be a random variable such that $P(X \neq 0) > 0$.
Then there exists $\varepsilon > 0$ such that $P(|X| > \varepsilon) > 0$.

Proof: Assume the contrary, i.e. $P(|X| \leq \varepsilon) = 1$ for all $\varepsilon > 0$.

$$P\left(|X| \leq \frac{1}{n}\right) = 1 \quad \text{for all } n \in \mathbb{N}.$$

Note that $\{X = 0\} = \bigcap_{n \in \mathbb{N}} \{|X| \leq \frac{1}{n}\}$

Therefore, by the continuity property,

$$P(X=0) = \lim_n P\left(|X| \leq \frac{1}{n}\right) = \lim_n 1 = 1.$$

Contradiction completes the proof of the Claim.

Now let $\varepsilon > 0$ be as in the Claim for X_1. Hence

$$\sum_{n=1}^{\infty} P(|X_n| > \varepsilon) = \sum_{n=1}^{\infty} P(|X_1| > \varepsilon) \quad \text{by the identical distribution}$$

$$= \infty$$

By independence, Borel-Cantelli lemma II yields:

$$P(|X_n| > \varepsilon \text{ i.o.}) = 1$$

Since $|X_n| > \varepsilon$ i.o. implies that $\sum_n X_n$ does not converge, we conclude:

$$P\left(\sum_n X_n \text{ does not converge}\right) = 1.$$

QED.

(5)

Denote by X_n the length of the remaining part of the stick after n steps. Then X_n is a random variable, and

$$Y_1 := X_1 \quad \text{is uniform in } (0,1)$$

$$Y_2 := \frac{X_2}{X_1} \quad \text{is uniform in } (0,1)$$

$$\cdots\cdots\cdots$$

$$Y_n := \frac{X_n}{X_{n-1}} \quad \text{is uniform in } (0,1)$$

$$\cdots\cdots$$

Furthermore, since the steps are independent, Y_n are independent, and identically distributed. Multiplying the identities above, we see that

$$X_n = Y_1 \cdots Y_n .$$

Then $$\frac{1}{n}\log X_n = \frac{1}{n}\sum_{k=1}^{n}\log Y_k.$$

$$\mathbb{E}\log Y_1 = \int_0^1 \log y\,dy = (y\log y - y)\Big|_0^1 = -1.$$

By the Strong Law of Large Numbers,

$$\frac{1}{n}\log X_n \to -1 \quad \text{almost surely as } n\to\infty$$

Exponentiating, and noting that $f(x)=e^x$ is continuous thus preserves pointwise convergence we have

$$X_n^{1/n} \to e^{-1} \quad \text{almost surely as } n\to\infty.$$

QED.

(a) By translation, we can assume that $\mathbb{E} X_n = 0$.

It is enough to prove that $\mathbb{E}|S_n| \to 0$ as $n \to \infty$ (Proposition 15.3)

First consider the case where n is an even number; $n = 2m$, $m \in \mathbb{N}$.

We can decompose S_n into the sum of even and odd terms:

$$S_n = \frac{1}{2}(U_m + V_m) \quad \text{where}$$

$$U_m = \frac{1}{m} \sum_{k=1}^{m} X_{2k} \ ; \qquad V_m = \frac{1}{m} \sum_{k=1}^{m} X_{2k-1}.$$

By the assumption, the terms in U_m ~~and the~~ are independent, and similarly for V_m. Hence the Weak Law of Large Numbers applies to U_m and to V_m (separately).

In the form we proved WLLN, it states

$$\mathbb{E}|U_m| \to 0 \quad \text{and} \quad \mathbb{E}|V_m| \to 0 \qquad \text{as } m \to \infty.$$

(Note: we proved $\mathbb{E}|U_m|^2 \to 0$, but Hölder's inequality says $\mathbb{E}|U_m| \le (\mathbb{E}|U_m|^2)^{1/2} \to 0$, see Proposition 14.5)

Therefore,

$$\mathbb{E}|S_n| \le \frac{1}{2}\,\mathbb{E}|U_m + V_m| \le \frac{1}{2}\left(\mathbb{E}|U_m| + \mathbb{E}|V_m|\right) \to 0$$

as $n \to \infty$.

Finally, if n is odd, then there will be an extra term of the form $\mathbb{E}\left|\frac{X_n}{n}\right| \le \frac{1}{n} \to 0$ (Exercise: provide details).

QED

(6) We can again assume that $\mathbb{E}X_k = 0$.

By Proposition 15.3 in the lecture notes, it suffices to prove that

$$\mathbb{E}(S_n - \mathbb{E}S_n)^2 = \mathbb{E}S_n^2 \to 0 \quad \text{as } n \to \infty.$$

$$\mathbb{E}S_n^2 = \frac{1}{n^2}\,\mathbb{E}\Big(\sum_{k=1}^{n} X_k^2\Big) = \frac{2}{n^2}\sum_{k \geq j} \mathbb{E}X_j X_k = \frac{2}{n^2}\sum_{k\geq j} \mathrm{Cov}(X_j, X_k) \qquad (\square)$$

Let $\varepsilon > 0$. By the assumption, $\exists\, N_\varepsilon$ s.t.

$$\mathrm{Cov}(X_j, X_k) < \varepsilon \quad \text{for all } k, j : \ k - j > N_\varepsilon . \qquad (☺)$$

Fix any $n > N_\varepsilon$, and write

$$\sum_{k \geq j} \mathrm{Cov}(X_j, X_k) = \sum_{\substack{0 \leq k-j \leq N_\varepsilon \\ 1 \leq k,j \leq n}} \mathrm{Cov}(X_j, X_k) + \sum_{\substack{k-j > N_\varepsilon \\ 1 \leq k,j \leq n}} \mathrm{Cov}(X_j, X_k) \qquad (*)$$

The first sum in $(*)$ has at most $n N_\varepsilon$ terms. Every term is bounded by

$$\mathrm{Cov}(X_j, X_k) = \mathbb{E}X_j X_k \leq \left(\mathbb{E}X_j^2\right)^{1/2}\left(\mathbb{E}X_k^2\right)^{1/2} \quad \text{by Cauchy-Schwarz (Corollary 14.7)}$$

$$= \mathrm{Var}(X_1) =: C \quad \text{by the identical distribution.}$$

Hence the first sum in $(*)$ is bounded by

$$n N_\varepsilon C.$$

The second sum in $(*)$ has at most n^2 terms, where each term is bounded by ε (by ☺). Hence this sum is bounded by

$$n^2 \varepsilon.$$

Therefore,

$$\sum_{k \geq j} \mathrm{Cov}(X_j, X_k) \leq n N_\varepsilon C + n^2 \varepsilon,$$

so by $(\square)$

$$\mathbb{E}S_n^2 \leq \frac{2 C N_\varepsilon}{n} + 2\varepsilon$$

Let $N'_\varepsilon := \frac{C N_\varepsilon}{\varepsilon}$. Then for $n > N'_\varepsilon$ we have

$$\mathbb{E} S_n^2 < 2\varepsilon + 2\varepsilon \leq 4\varepsilon.$$

This shows that

$$\mathbb{E} S_n^2 \to 0 \quad \text{as} \quad n \to \infty. \qquad \text{Q.E.D.}$$