**Final Exam (2 pages)**
Probability Theory (MATH 235A, Fall 2007)

**1. (12 pts)** One tosses a fair coin (H=Heads, T=Tails) until the first appearance of the pattern HT. How many tosses on average is made?

**2. (13 pts)** Prove that, for a random variable $X$ with finite variance, one has

$$\mathrm{Var}(X) = \min_{a \in \mathbb{R}} \mathbb{E}(X-a)^2.$$

**3.** Consider independent random variables $X_1, X_2, \dots$ uniformly distributed in $[0,1]$, and let $Z_n = \max_{k \le n} X_k$.

(a) **(8 pts)** Prove that $Z_n \to 1$ in probability.

(b) **(8 pts)** Prove that $Z_n \to 1$ almost surely.

**4.** Consider independent and identically distributed random variables $X_1, X_2, \dots$ such that $\mathbb{P}(X_k = 0) < 1$.

(a) **(11 pts)** Assuming that $X_k$ have finite mean, prove that

$$\mathbb{P}\Big(\sum_k X_k \text{ converges}\Big) = 0.$$

(b) **(16 pts)** Prove the same conclusion *without* the finite mean assumption.

**5. (16 pts)** A horizontal stick of length 1 is broken at a random point that is uniformly distributed. The right hand part is thrown away, and the left part is broken similarly (at a random point uniformly distributed in this part). Show that after $n$ steps, the remaining part has exponentially small length. More precisely, prove that its length $X_n$ satisfies

$$X_n^{1/n} \to 1/e \quad \text{almost surely.}$$

*(Hint: represent $X_n$ as a product of some random variables, and take logarithm).*

**6.** We say that random variables $X_1, X_2, \ldots$ with finite means satisfy the Weak Law of Large Numbers if their arithmetic means $S_n = \frac{1}{n}\sum_{k=1}^{n} X_k$ satisfy

$$S_n \to \mathbb{E}S_n \quad \text{in probability.}$$

(a) **(16 pts)** Consider random variables $X_1, X_2, \ldots$ with uniformly bounded variances: $\mathrm{Var}(X_k) \le C < \infty$ for all $k$. Suppose that $X_j, X_k$ are independent if $k > j+1$ (in other words, every $X_k$ depends only on $X_{k-1}$). Prove that these random variables satisfy the Weak Law of Large Numbers.

(b) **Bonus Problem, full credit only (15 pts)** The *covariance* of random variables $X, Y$ is defined as

$$\mathrm{Cov}(X,Y) = \mathbb{E}(X - \mathbb{E}X)(Y - \mathbb{E}Y).$$

Consider identically distributed random variables $X_k$ with finite variances. Suppose that

$$\mathrm{Cov}(X_j, X_k) \to 0 \quad \text{as } |j-k| \to \infty.$$

Prove these random variables satisfy the Weak Law of Large Numbers.