Homework 1.

(2)

Statement 1 : TRUE.

In words: $A_n \cup B_n$ occur infinitely often $\Longleftrightarrow$ either A_n occurs infinitely often or B_n occurs infinitely often.

This is true.

Mathematically, one verifies the definition:

$$\limsup_n (A_n \cup B_n) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} (A_k \cup B_k)$$

$$= \bigcap_{n=1}^{\infty} \left[\left(\bigcup_{k=n}^{\infty} A_k \right) \cup \left(\bigcup_{k=n}^{\infty} B_k \right) \right]$$

$$= \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \right) \cup \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B_k \right)$$

$$= \limsup_n A_n \cup \limsup_n B_n .$$

Statement 2 : FALSE.

In words:

Both A_n and B_n occur infinitely often $\Longleftrightarrow$
A_n occurs infinitely often <u>AND</u> B_n occurs infinitely often.

"$\Leftarrow$" is not true : A_n can occur for <u>even</u> n,
B_n can occur for <u>odd</u> n.

Both A_n and B_n will never occur.

Mathematically, a counterexample would be:

$$A_n = \begin{cases} 1 & \text{for } n \text{ even} \\ \emptyset & \text{for } n \text{ odd} \end{cases}$$

$$B_n = \begin{cases} 1 & \text{for } n \text{ odd} \\ \emptyset & \text{for } n \text{ even}. \end{cases}$$

Then $\limsup_n (A_n \cap B_n) = \limsup_n (\emptyset) = \emptyset$,

while $\limsup A_n = \{1\}$, $\limsup B_n = \{1\}$.

<u>Note</u>: "$\Rightarrow$" is true:

$$\limsup_n (A_n \cap B_n) \subseteq \limsup_n A_n \cap \limsup_n B_n$$

(Exercise: prove this)

<u>Statement 3</u>: FALSE.

Argument is similar. You can also take complements and deduce it formally from the previous proof.

<u>Note</u>: It is true that

$$\boxed{\liminf_n (A_n \cup B_n) \supseteq \liminf_n A_n \cup \liminf_n B_n,} \qquad (*)$$

(Exercise: prove this)

<u>Statement 4</u>: TRUE.

Either take complements and deduce from Statement 1, or argue similarly to the proof of Statement 1.

Statement 5 :

By inclusion (*) on p. 2,

$$(\#) \qquad \liminf_n (A_n \cup B_n) \supseteq \liminf_n A_n \cup \liminf_n B_n = A \cup B.$$

By Statement 1,

$$(\#\#) \qquad \limsup_n (A_n \cup B_n) = \limsup_n A_n \cup \limsup_n B_n = A \cup B.$$

Since $\liminf (A_n \cup B_n) \subseteq \limsup (A_n \cup B_n)$, it follows that

(#) and (##) are the same set $A \cup B$.

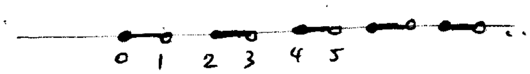
In particular, $\limsup_n (A_n \cup B_n) = \liminf_n (A_n \cup B_n)$ QED

(3)

Fix an $n \in \mathbb{N}$. Every element A of $\mathcal{A}_n$ is a subset of $\mathbb{R}$ with the following property: either A or A^c contains a half-infinite interval of the form $(N, +\infty)$.

Therefore, every element A of $\mathcal{A} := \bigcup_n \mathcal{A}_n$ has this property, too.

Suppose $\mathcal{A}$ is a σ-algebra. Since $\mathcal{A}$ contains (as elements) all intervals of the form $[k-1, k)$, it must also contain (as a σ-algebra) the set $A := [0,1) \cup [2,3) \cup [4,5) \cup \ldots$



However, neither A nor A^c contains a half-infinite interval.

Contradiction. QED

④

The inclusion

$$\sigma(B_1, B_2 \dots B_n) \subseteq \sigma(A_1, A_2 \dots A_n)$$

is obvious because B_k's are finite unions of A_k's.

The converse inclusion will ~~b~~ also be obvious if we prove that A_k's are finite ~~unions~~ intersections of B_k's or their complements. The latter is indeed true:

$$A_1 = B_1$$

$$A_2 = B_2 \setminus B_1 = B_2 \cap B_1^c$$

$$A_3 = B_3 \setminus B_2 = B_3 \cap B_2^c$$

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QED.

⑦

The conclusion follows at once from these two identities:

$$I(\limsup A_n) = \limsup I(A_n),$$

$$I(\liminf A_n) = \liminf I(A_n),$$

which hold pointwise (i.e. for values of the indicator function at every $x \in \mathbb{R}$).

To prove the first identity, note that for every sets $B_1, B_2, \dots$

$$I\left(\bigcup_n B_n\right) = \sup_n I(B_n), \qquad I\left(\bigcap_n B_n\right) = \inf_n I(B_n). \qquad (*)$$

(this is easy because ~~the~~ the indicator function takes only values 0 and 1)

Using this, we ~~have~~ get

$$I(\limsup_n A_n) = I\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right)$$

$$= \inf_n I\left(\bigcup_{k=n}^{\infty} A_k\right)$$ by the second identity in (*) on p. 4, used for $B_n = \bigcup_{k\geq n}^{\infty} A_k$

$$= \inf_n \sup_{k\geq n} I(A_k)$$ by the first identity in (*) on p. 4

$$= \liminf_n I(A_n).$$

The second identity in Problem 7 is analogous. QED.

9

First inequality:

For $n=2$ we know from the subadditivity property that the inequality is true: $P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$.

For general n, use induction to complete the proof.

Second inequality. For $n=2$, it is not hard to prove this equality.

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2). \qquad (*)$$

For general n, use induction as follows.

Suppose we proved the inequality for up to $n-1$.

Let $B_{n-1} = \bigcup_{k=1}^{n-1} A_k$, and $C_n = B_n \cap A_n = \bigcup_{k=1}^{n-1} (A_k \cap A_n)$.

Then $P\left(\bigcup_{k=1}^{n} A_k\right) = P(B_{n-1} \cup A_n) = P(B_{n-1}) + P(A_n) - P(C_n)$

by (*)

Now, $\mathbb{P}(B_{n-1})$ is estimated from below by the induction hypothesis:

$$\mathbb{P}(B_{n-1}) \geq \sum_{k=1}^{n-1} \mathbb{P}(A_k) - \sum_{1 \leq i < j \leq n-1} \mathbb{P}(A_i \cap A_j)$$

and $\mathbb{P}(C_n)$ from above by $(*)$:

$$\mathbb{P}(C_n) \leq \sum_{k=1}^{n-1} \mathbb{P}(A_k \cap A_n)$$

Joining these estimates yields the conclusion. QED.