Homework 2. Solutions.

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a) Consider $B_n = A_n^c$. Then $P(B_n) \to 0$, and we want to show that $\exists$ a subsequence (n_k) such that
$$P\left(\bigcup_{k=1}^{\infty} B_{n_k}\right) < 1. \qquad (*)$$

(From this, the original conclusion about A_n's follows by de Morgan formulas).

To prove $(*)$, we ~~can~~ select a subsequence such that
$$P(B_{n_k}) \le \frac{1}{2^{k+1}}, \quad k = 1, 2, \ldots$$
(there are other choices as well).

Then
$$P\left(\bigcup_{k=1}^{\infty} B_{n_k}\right) \le \sum_{k=1}^{\infty} P(B_{n_k}) \le \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2}.$$

We have proved $(*)$. QED.

b) Let (A_n) be independent events, say with $P(A_n) = \frac{1}{2}$.
For example, A_n = "Heads in n'th toss of a fair coin".
Then, for every K:
$$P\left(\bigcap_{k=1}^{K} A_{n_k}\right) = \prod_{k=1}^{K} P(A_{n_k}) = \frac{1}{2^K} \to 0 \quad \text{as } K \to \infty.$$
Hence by the continuity (Thm. 3.2), $P\left(\bigcap_{k=1}^{\infty} A_{n_k}\right) = 0$.
This completes the proof of (b).

c) Since the sets A_n are non-increasing,

$$\bigcap_{n=1}^{\infty} A_n = \lim_n A_n.$$

By the continuity property and by the assumption $P(A_n) \geq \alpha$,

$$P(\lim_n A_n) = \lim_n P(A_n) \geq \alpha.$$

This completes the proof.

d). Similar to (c).

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$$P\left(\bigcup_{k=1}^{n} A_k\right) = 1 - P\left(\left(\bigcup_{k=1}^{n} A_k\right)^c\right)$$

$$= 1 - P\left(\bigcap_{k=1}^{n} A_k^c\right) \quad \text{by de Morgan}$$

$$= 1 - \prod_{k=1}^{n} P(A_k^c) \quad \text{by the independence}$$

$$= 1 - \prod_{k=1}^{n} (1 - P(A_k)).$$

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Recall the inequality $\log(1-x) \leq -x$ for $x \in (0,1)$.

Then by Problem 11,

$$\log\left(1 - P\left(\bigcup_{k=1}^{n} A_k\right)\right) = \log\left(\prod_{k=1}^{n} (1 - P(A_k))\right) = \sum_{k=1}^{n} \log(1 - P(A_k)) \leq -\sum_{k=1}^{n} P(A_k).$$

This proves the first claim of the Problem.

The second claim follows by the continuity property.

We can read the conclusion as:

$$\text{"there } \underbrace{\text{exists}}_{\text{a point}} \ \omega \in \limsup_n A_n\text{"}$$

which is equivalent to

$$\limsup_n A_n \neq \emptyset. \qquad (*)$$

To prove $(*)$ it suffices to check that $\lambda(\limsup_n A_n) > 0$.

This follows from Theorem 3.2 (i) and the assumption. Indeed,

$$\lambda(\limsup_n A_n) \geqslant \limsup_n \lambda(A_n) \geqslant \eta.$$

The proof is complete.