Consider $B_n = A_n^c$. Then $P(B_n) \to 0$, and we want to show that there exists a subsequence $(n_k)$ such that

$$P\left(\bigcup_{k=1}^{\infty} B_{n_k}\right) < 1. \quad (*)$$

(From this, the original conclusion about $A_n$'s follows by de Morgan formulas).

To prove $(*)$, we need to select a subsequence such that

$$P(B_{n_k}) \leq \frac{1}{2^k+1}, \quad k=1,2,\ldots.$$

(There are other choices as well).

Then

$$P\left(\bigcup_{k=1}^{\infty} B_{n_k}\right) \leq \sum_{k=1}^{\infty} P(B_{n_k}) \leq \sum_{k=1}^{\infty} \frac{1}{2^k+1} = \frac{1}{2}.$$ We have proved $(*)$. QED.

b) Let $(A_n)$ be independent events, as say with $P(A_n) = \frac{1}{2}$. For example, $A_n$ = "Head in $n$th toss of a fair coin." Then, for every $K$:

$$P\left(\bigcap_{k=1}^{K} A_{n_k}\right) = \prod_{k=1}^{K} P(A_{n_k}) = \frac{1}{2^K} \to 0 \text{ as } K \to \infty.$$ Hence by the continuity (Thm. 3.2),

$$P\left(\bigcap_{k=1}^{\infty} A_{n_k}\right) = 0.$$ This completes the proof of (b).
c) Since the sets $A_n$ are non-increasing,
\[ \bigcap_{n=1}^{\infty} A_n = \lim_{n \to \infty} A_n. \]
By the continuity property and by the assumption $P(A_n) > \alpha$,
\[ P\left( \lim_{n \to \infty} A_n \right) = \lim_{n \to \infty} P(A_n) \geq \alpha. \]
This completes the proof.

d) Similar to (c).

\[ \prod \]

\[ P\left( \bigcup_{k=1}^{\infty} A_k \right) = 1 - P\left( (\bigcup_{k=1}^{\infty} A_k)^c \right) \]
\[ = 1 - P\left( \bigcap_{k=1}^{\infty} A_k^c \right) \quad \text{by de Morgan} \]
\[ = 1 - \prod_{k=1}^{\infty} P(A_k^c) \quad \text{by the independence} \]
\[ = 1 - \prod_{k=1}^{\infty} (1 - P(A_k)). \]

\[ \text{(12)} \]
Recall the inequality $\log(1 - x) \leq -x$ for $x \in (0, 1)$.
Then by Problem 11,
\[ \log\left(1 - P\left( \bigcup_{k=1}^{\infty} A_k \right) \right) = \log\left( \prod_{k=1}^{\infty} (1 - P(A_k)) \right) = \sum_{k=1}^{\infty} \log(1 - P(A_k)) \leq -\sum_{k=1}^{\infty} P(A_k). \]
This proves the first claim of the problem.

The second claim follows by the continuity property.
We can read the conclusion as:

"there exists $\omega \in \limsup A_n$"

which is equivalent to

$$\limsup A_n \neq \emptyset. \quad (\ast)$$

To prove (\ast) it suffices to check that $\lambda(\limsup A_n) > 0$.

This follows from Theorem 3.2 (i) and the assumption. Indeed,

$$\lambda(\limsup A_n) \geq \limsup \lambda(A_n) \geq \eta.$$

The proof is complete.