Homework 3 Solutions.

(1)

In words, $\limsup A_n$ = "A_n occur infinitely often",

$\liminf A_n$ = "all except finitely many of A_n occur"

These events do not depend on whether finitely many of A_n's occur or not, therefore they are tail events.
Kolmogorov's zero-one law finishes the proof.

To prove rigorously that $\limsup A_n$ is a tail event, write it as

$$\limsup A_n = \bigcap_{n=1}^{\infty} B_n, \text{ where } B_n = \bigcup_{k=n}^{\infty} A_k.$$

B_k are non-increasing $\Rightarrow$ $\bigcap_{n=1}^{\infty} B_n = \bigcap_{n=n_0}^{\infty} B_n$ for every $n_0 \geq 1$.

Hence $\limsup A_n = \bigcap_{n=n_0}^{\infty} \bigcup_{k=n}^{\infty} A_k \in \sigma(A_{n_0}, A_{n_0+1}, \dots)$.

Therefore, $\limsup A_n$ is a tail event.

The proof for $\liminf A_n$ is similar.

2

It is enough to show that the probabilities of the events

$$A_n = \{|X| > n\}$$

converge to zero as $n \to \infty$.

To prove this, we use Theorem 3.2 (i) and see that

$$\limsup \mathbb{P}(A_n) \le \mathbb{P}(\limsup A_n) = \mathbb{P}(|X| = \infty) = 0.$$

Q.E.D.

3

Using Problem 2, choose K so that

$$\mathbb{P}(|X| > K) < \varepsilon.$$

Define X_ε as follows:

$$X_\varepsilon = \begin{cases} X, & \text{when } |X| \le K \\ 0 & \text{otherwise} \end{cases} = X \cdot \mathbb{1}_{\{|X| \le K\}}.$$

Then X is a random variable, and it is bounded by construction.

Finally,
$$\mathbb{P}(X \ne X_\varepsilon) = \mathbb{P}(|X| > K) < \varepsilon.$$

Q.E.D.

4

For every $a \in \mathbb{R}$,

$$\{|X| \le a\} = \{X \le a\} \cap \{X \ge -a\}$$

Both $\{X \le a\}$ and $\{X \ge -a\}$ are measurable because X is a random variable; so is ~~the~~ their intersection.

5

There are 2^4 possible ways to choose who of the four children are boys and who are girls.
For each k, there are $\binom{4}{k}$ ways to make such a choice if exactly k children need to be boys.
The probability of each choice is the same $\Rightarrow$

$$\mathbb{P}(k \text{ boys}) = \frac{\binom{4}{k}}{2^4}.$$

This is maximal when $k=2$ $\Rightarrow$ the most likely family has 2 boys and 2 girls. The corresponding probability is

$$\binom{4}{2} \cdot 2^{-4} = \frac{3}{8}.$$

6

Let $f : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ and $g : (S, \mathcal{S}) \to (T, \mathcal{T})$ be measurable maps. Consider $B \in \mathcal{T}$. Then

$$\{\omega : g(f(\omega)) \in B\} = \{\omega : f(\omega) \in g^{-1}(B)\} \in \mathcal{F},$$

since by the assumption we have $g^{-1}(B) \in \mathcal{S}$. QED.

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This is Theorem 1.1 on p. 26.