Homework 3

Probability Theory (MATH 235A, Fall 2007)

1. Zero-one law.

Let $A_1, A_2, \ldots$ be independent events. Prove that the events $\limsup_n A_n$ and $\liminf_n A_n$ each have probabilities either zero or one.

2. Bounding random variables.

Let X be a random variable. Prove that for every $\varepsilon > 0$ there exists K such that

$$\mathbb{P}(|X| > K) < \varepsilon.$$

(Hint: Using Theorem 3.2, show that the probabilities of the events $A_n = \{|X| > n\}$ converge to zero as $n \to \infty$.)

3. Approximation by bounded random variables.

Let X be a random variable. Show that for every $\varepsilon > 0$ there exists a bounded random variable X_ε such that

$$\mathbb{P}(X \neq X_\varepsilon) < \varepsilon.$$

(Hint: use Problem 2).

4. Absolute value

Show that if X is a random variable then so is $|X|$.

5. Family

There are 4 children in a family. The probabilities of a boy or a girl are both 1/2. Find the most probable number of boys and girls in the family, and the corresponding probability.

6. Composition of measurable maps.

Prove that a composition of two measurable maps is measurable.

7. Random variables induce probability measures on $\mathbb{R}$

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that X induces a probability measure on $\mathbb{R}$ in the following sense. For every Borel subset A of $\mathbb{R}$, define

$$P(A) := \mathbb{P}(X \in A) = \mathbb{P}(\omega \in \Omega : \; X(\omega) \in A).$$

Prove that $(\mathbb{R}, \mathcal{R}, P)$ is a probability space.