Homework 4 Solutions.

①

It suffices to show that

$$\{\omega : X(\omega) \le x\} = \{\omega : \omega \le F(x)\}. \qquad (*)$$

Indeed, if this holds then

$$\mathbb{P}\{\omega : X(\omega) \le x\} = \mathbb{P}\{\omega : \omega \le F(x)\}$$
$$= \mathbb{P}\big([0, F(x)]\big) = F(x)$$

because $\mathbb{P}$ is Lebesgue measure, and we are done.

To check $(*)$, we prove two inclusions:

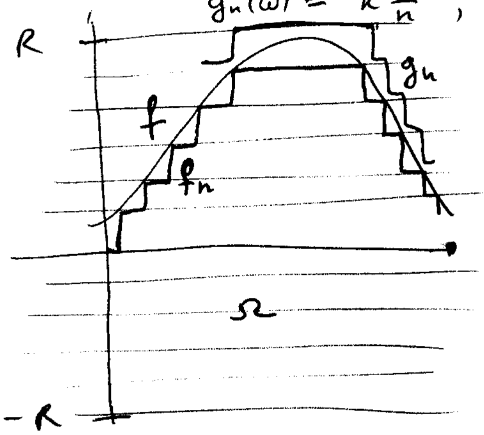
"$\supseteq$": let $\omega \le F(x)$. We want to show that $X(\omega) \le x$.
This amounts to proving that for every y such that $F(y) < \omega$, we have $y \le x$.
For every such y, $F(y) < \omega \le F(x)$. Since F is nondecreasing, $y \le x$ follows.

"$\subseteq$": We argue by contraposition. Assume $X(\omega) \le x$ but $\omega > F(x)$. Since F is right-continuous, $\exists\, \varepsilon > 0$ such that $F(x+\varepsilon) < \omega$. This means (by the definition of X) that
$X(\omega) \ge x + \varepsilon > x$.
The contradiction completes the proof.

We define, for a function f, $|f| \le R$, a sequence of simple functions

$$f_n(\omega) = (k-1)\frac{R}{n} \qquad \text{for } (k-1)\frac{R}{n} < f(\omega) < k\frac{R}{n}, \qquad k = 1, 2, \dots n.$$

$$g_n(\omega) = k\frac{R}{n}, \quad \text{"} \quad \text{"}$$



(a) Then, for every ω,

$$0 \le f(\omega) - f_n(\omega) \le \frac{R}{n} \to 0 \text{ as } n \to \infty. \qquad (*)$$

$$0 \le g_n(\omega) - f(\omega) \le \frac{R}{n}$$

This proves $f_n \to f$ pointwise.

(b) We want now to prove

$$\sup_{\phi \le f} \int \phi \, d\lambda = \inf_{\psi \ge f} \int \psi \, d\lambda, \qquad \phi, \psi \text{ simple}.$$

The inequality "$\le$" follows from the monotonicity property:

$\phi \le f \le \psi$ implies $\int \phi \, d\lambda \le \int \psi \, d\lambda$.

The converse inequality is equivalent to the following statement: for every ε there exist simple functions ϕ, ψ such that

$$\phi \le f \le \psi \quad \text{and}$$

$$\int \phi \, d\lambda > \int \psi \, d\lambda - \varepsilon.$$

This is immediate from $(*)$ in part (a) and the properties of the integral: choose n big enough so that $\frac{2R}{n} < \varepsilon$; then for $\phi = f_n$, $\psi = g_n$ $\Rightarrow$

$$0 \le \int \psi \, d\lambda - \int \phi \, d\lambda = \int (\psi - \phi) \, d\lambda \le \frac{2R}{n} < \varepsilon.$$

QED.

a) The density of a random variable is the derivative of its distribution function.

Let $F_{\phi(X)}$, F_X denote the distribution functions of $\phi(X)$ and X respectively.

Then

$$F_{\phi(X)}(x) = P(\phi(X) \le x) = P(X \le \phi^{-1}(x)) = F_X(\phi^{-1}(x)).$$

Therefore the density of $\phi(X)$ is

$$F'_{\phi(X)}(x) = F'_X(\phi^{-1}(x))\,(\phi^{-1}(x))' = \boxed{\frac{f(\phi^{-1}(x))}{\phi'(\phi^{-1}(x))}}.$$

b) $\phi^{-1}(x) = \dfrac{x-b}{a}$; $\phi'(x) = a$, hence the density of $aX+b$ is

$$\boxed{\frac{1}{a} f\left(\frac{x-b}{a}\right)}$$

c) $F_{X^2}(x) = P(X^2 \le x) = P(-\sqrt{x} \le X \le \sqrt{x}) = F_X(\sqrt{x}) - F_X(-\sqrt{x}).$

Differentiating, we get

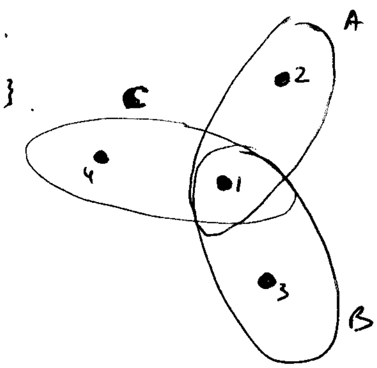
$$\frac{d}{dx} F_X(\sqrt{x}) = \frac{f(\sqrt{x})}{2\sqrt{x}}, \qquad \frac{d}{dx} F_X(-\sqrt{x}) = -\frac{f(-\sqrt{x})}{2\sqrt{x}}.$$

Thus the density of X^2 is

$$\boxed{\frac{f(\sqrt{x}) + f(-\sqrt{x})}{2\sqrt{x}}}.$$

d) For $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, we get the density of X^2 equal to

$$\boxed{\frac{1}{\sqrt{2\pi x}} e^{-x/2}} \quad \leftarrow \chi^2\text{-distribution}.$$



Let $\Omega = \{1, 2, 3, 4\}$, $\mathbb{P}$ = uniform distribution.

$A := \{1,2\}$, $B := \{1,3\}$, $C := \{1, 4\}$.

Then $\mathbb{P}(A) = \frac{1}{2}$, $\mathbb{P}(B) = \frac{1}{2}$, $\mathbb{P}(C) = \frac{1}{2}$.

Now, $\mathbb{P}(A \cap B) = 1/4 \Rightarrow$ A, B are independent

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But
$$\left\{\begin{aligned} \mathbb{P}(A \cup B) &= \frac{3}{4} \\ \mathbb{P}(C) &= \frac{1}{2} \end{aligned}\right\}, \quad \mathbb{P}\big((A \cup B) \cap C\big) = \frac{1}{4}$$

$$\frac{3}{4} \cdot \frac{1}{2} \neq \frac{1}{4} \Rightarrow A \cup B \text{ is not independent of } C.$$

A, B, C are not independent (otherwise, by a σ-algebra theorem, $A \cup B$ would be independent of C)

1) Let $|\Omega| = n$. The maximum number of events is 2^n (all subsets). ~~Clearly~~ Moreover we can make all 2^n subsets of Ω independent by defining a probability measure $\mathbb{P}$ that satisfies

$$\mathbb{P}(\omega) = 1 \quad \text{for some point } \omega \in \Omega.$$

(It is easy to check that all $\boxed{2^n}$ subsets are independent).

2) But there is more to this problem. Suppose we want to consider only probability measures for which

$$0 < \mathbb{P}(A) < 1$$

for every proper $A \subset \Omega$.

Then there can be at most $\log_2 n$ independent events.

Let's prove this. Suppose $A_1, \dots, A_m$ are independent and nonempty. Then all sets of the form

$$A_1^{(c)} \cap A_2^{(c)} \cap A_3^{(c)} \cap \dots \cap A_m^{(c)},$$

where we put or don't put c (the complement) for each A_k's, are disjoint and non-empty.

(Indeed, every two such sets must differ in at least one place where one has A_k while the other has A_k^c; these must be disjoint. Also, by the independence, each set has positive measure $\Rightarrow$ non-empty.

~~Therefore,~~ The number of such sets is 2^m (choose whether to put or not put c in 2^m ways).

Since they are disjoint, $n = |\Omega| \geq 2^m$.

Hence $\boxed{m \leq \log_2 n}$

(May be $+1$ for the empty set).

<u>This bound is attained</u>: let $\Omega = \{H, T\}^m$ where $m = \log_2 n$ for some n. Then $|\Omega| = n$.

Let $A_k = \{ H \text{ in } k\text{'th toss of a coin} \}$, $k = 1, \dots, m$

These are independent.

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$$P(X < y + x \mid X \geq y) = \frac{P(y \leq X < y + x)}{P(X \geq y)} = \frac{F(x+y) - F(y)}{1 - F(y)}.$$

By the assumption, the LHS $= F(x)$.

Let $\Phi(x) = 1 - F(x)$; then we get

$$\frac{\Phi(y) - \Phi(x+y)}{\Phi(y)} = 1 - \Phi(x).$$

from which it follows that

$$\Phi(x+y) = \Phi(x) \cdot \Phi(y)$$

Taking logarithms we see that

$$(\log \Phi)(x+y) = (\log \Phi)(x) + (\log \Phi)(y) \qquad \forall x, y \geq 0$$

Which means that $\log \Phi$ is a linear function:

$$(\log \Phi)(x) = ax + b.$$

Hence $\Phi(x) = e^{ax+b}$, $x \geq 0$

so $F(x) = 1 - e^{ax+b}$

The condition ~~F(0) = 0~~ $X \geq 0$ means $F(0) = 0 \Rightarrow b = 0$

The condition $F(x) \to 1$ as $x \to \infty$ means $a < 0 \Rightarrow$

$a = -\lambda$ for some $\lambda > 0$.

Therefore $\boxed{F(x) = 1 - e^{-\lambda x}}$.