Homework 4

Probability Theory (MATH 235A, Fall 2007)

1. Generating random variables with given distributions. Consider a funciton $F : \mathbb{R} \to \mathbb{R}$ that satisfies:

(i) F is nondecreasing and $0 \le F(x) \le 1$ for all x;

(ii) $F(x) \to 0$ as $x \to -\infty$ and $F(x) \to 1$ as $x \to \infty$;

(iii) F is right continuous.

Prove that F is a distrubution function of some random variable X. To do so, consider the probability space $(\Omega, \mathcal{R}, \mathbb{P})$ where $\Omega = [0, 1]$, $\mathcal{R}$ is the Borel σ-algebra, and $\mathbb{P}$ is the Lebesgue measure. Define X by the assignment

$$X(\omega) := \sup\{y : \ F(y) < \omega\}$$

and prove that X is a random variable with distribution function F

2. Approximation by simple functions. a) Let f be a bounded measurable function. Prove that there exists a sequence of simple functions f_n such that $f_n \to f$ pointwise. (Hint: see the proof of Lemma 2.1.)

b) Refer to Definition 2.5 of Lebesgue integral of simple functions. Prove that for every bounded measurable function f, we have

$$\sup_{\phi \le f} \int \phi \, d\lambda = \inf_{\psi \ge f} \int \psi \, d\lambda,$$

where ϕ and ψ are simple functions.

3. Functions of random variables. Let X be a random variable with continuous density f. Let ϕ be a strictly increasing and differentiable function.

a) Compute the density of the random variable $\phi(X)$.

b) What is this density for the linear function $\phi(x) = ax + b$?

c) What is this density for the function $\phi(x) = x^2$?

d) Work out the answer when X has a standard normal distribution; X^2 is called the *chi-square distribution.*

4. Pairwise independence. Give an example of three events A, B and C such that A is independent of B, B is independent of C and A is independent of C, but $A \cup B$ is not independent of C. Deduce that A,B,C are not independent.

5. Independence needs a big sample space. How many independent events can be defined in the sample space that consists of n elements? Give a bound above for arbitrary probability measure on Ω, and show that this bound is sharp.

6. Exponential distribution. Let X be a non-negative random variable with absolutely continuous distribution. Assume that the conditional probability satisfies

$$\mathbb{P}(X < y + x \mid X \geq y) = \mathbb{P}(X < x), \qquad x \geq 0,\ y \geq 0.$$

Prove that X has an *exponential distribution*, i.e. its distribution function has the form

$$F(x) = 1 - e^{-\lambda x}, \qquad x \geq 0,$$

for some parameter $\lambda > 0$.