(1)

A definite statement could be this:
Suppose \( E \left( \sum_{n=1}^{\infty} X_n \right) < \infty \). Then \( \sum_{n=1}^{\infty} EX_n < \infty \) and the two quantities are equal.

Indeed, consider the random variables
\[ S_N = \sum_{n=1}^{N} X_n \] (partial sums).

Since \( X_n \geq 0 \), we have \( S_N \geq 0 \) and \( S_N / S := \sum_{n=1}^{\infty} X_n \). (\( N \to \infty \))

By the Monotone Convergence Theorem,
\[ \mathcal{E}S_N \to \mathcal{E}S \quad (N \to \infty) \]

\[ \sum_{n=1}^{\infty} \mathcal{E}X_n . \]

Q.E.D.

(2)

Let \( t \geq 1 \).
Let \( p \in (0,1) \) be a parameter to be chosen later, and define a r.v.
\[ X = \begin{cases} 1 & \text{with probability } \frac{p}{2} \\ -1 & \text{with probability } \frac{p}{2} \\ 0 & \text{with probability } 1-p \end{cases} \]

Then \( \mu = \mathcal{E}X = 0 \), \( \mathcal{E}X^2 = t^2 \frac{p}{2} + (-1)^2 \frac{p}{2} = p \), so
\[ \sigma^2 = \text{var}(X) = p . \]

Now, \( P \left( |X - \mu| \geq t \sigma \right) = P(X \neq 0) = p \)

To make the right hand side as in Chebyshev's inequality, we choose \( p = \frac{1}{1+t^2} \leq 1 \) (by the assumption) Q.E.D.
\[
\int R P(x < X < x+a) \, dx = \int R E \mathbb{1}_{\{X < x < x+a\}} \, dx \\
= E \int R \mathbb{1}_{\{X < x < x+a\}} (x) \, dx \quad \text{by Fubini} \\
= E a = a.
\]

First, we observe that,
\[
E|X - \frac{1}{2}| = \int_0^1 |x - \frac{1}{2}| \, dx = \frac{1}{4},
\]
and similarly \( E|Y - \frac{1}{2}| = \frac{1}{4} \).

Since by the triangle inequality,
\[
|X - Y| = |(X - \frac{1}{2}) - (Y - \frac{1}{2})| \leq |X - \frac{1}{2}| + |Y - \frac{1}{2}|,
\]
we have
\[
E|X - Y| \leq E|X - \frac{1}{2}| + E|Y - \frac{1}{2}| \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
\]
I will explain the solution for \( n=2 \); the general case is similar. So, let \( X, Y \) be random variables such that

\[
P \left( (X, Y) \in D \right) = \int_D f(x)g(y) \, dx \, dy \quad \text{for} \quad D \subseteq \mathbb{R}^2.
\]

Use then for \( D = A \times B \) by Fubini

\[
(x) \quad P(X \in A, Y \in B) = \int_A f(x)dx \cdot \int_B g(y)dy \quad \text{for} \quad A, B \subseteq \mathbb{R}.
\]

As we are told, \( f \) and \( g \) may not be densities, that is

\[a \int f = 1, \quad b \int g = 1,\]

but nevertheless for \( A = B = \mathbb{R} \) we get

\[
1 = P(X \in \mathbb{R}, Y \in \mathbb{R}) = ab.
\]

For \( B = \mathbb{R} \), we get from \((x)\).

\[
P(X \in A) = \int_A f(x)dx \cdot b;
\]

and for \( A = \mathbb{R} \), we get

\[
P(Y \in B) = \int_B g(y)dy \cdot a.
\]

Multiplying these two identities, and using \( ab = 1 \), we get

\[
P(X \in A)P(Y \in B) = \int_A f(x)dx \cdot \int_B g(y)dy.
\]

Together with \((x)\) this completes the proof.