HOMEWORK 5 SOLUTIONS

①

A definite statement could be this:

Suppose $\mathbb{E}\left(\sum_{n=1}^{\infty} X_n\right) < \infty$. Then $\sum_{n=1}^{\infty} \mathbb{E} X_n < \infty$ and the two quantities are equal.

Indeed, consider the random variables

$$S_N = \sum_{n=1}^{N} X_n \qquad \text{(partial sums)}.$$

Since $X_n \geq 0$, we have $S_N \geq 0$ and $S_N \nearrow S := \sum_{n=1}^{\infty} X_n$. $(N \to \infty)$

By the Monotone Convergence Theorem,

$$\begin{array}{ccc} \mathbb{E} S_N & \to & \mathbb{E} S \qquad (N \to \infty) \\ \| & & \\ \sum_{n=1}^{\infty} \mathbb{E} X_n . & & \end{array}$$

QED.

②

Let $t \geq 1$.

Let $p \in (0,1)$ be a parameter to be chosen later, and define a r.v.

$$X = \begin{cases} 1 & \text{with probability } p/2 \\ -1 & \text{with probability } p/2 \\ 0 & \text{with probability } 1-p \end{cases}.$$

Then $\mu = \mathbb{E}X = 0$, $\mathbb{E}X^2 = 1^2 p/2 + (-1)^2 p/2 = p$, so

$\sigma^2 = \text{var}(X) = p$.

Now, $\mathbb{P}(|X-\mu| \geq t\sigma) = \mathbb{P}(X \neq 0) = p$

To make the right hand side as in Chebychev's inequality, we choose $p := 1/t^2 \leq 1$ (by the assumption). Q.E.D.

③

$$\int_{\mathbb{R}} \mathbb{P}(x < X < x+a)\, dx = \int_{\mathbb{R}} \mathbb{E}\, \mathbb{1}_{\{x<X<x+a\}}\, dx$$

$$= \mathbb{E} \int_{\mathbb{R}} \mathbb{1}_{\{x<X<x+a\}}(x)\, dx \qquad \text{by Fubini}$$

$$= \mathbb{E}\, a = a.$$

④

First, we observe that

$$\mathbb{E}|X - \tfrac{1}{2}| = \int_0^1 |x - \tfrac{1}{2}|\, dx = \frac{1}{4},$$

and similarly $\mathbb{E}|Y - \frac{1}{2}| = \frac{1}{4}$.

Since by the triangle inequality,

$$|X-Y| = |(X-\tfrac{1}{2}) - (Y-\tfrac{1}{2})| \le |X-\tfrac{1}{2}| + |Y-\tfrac{1}{2}|,$$

we have

$$\mathbb{E}|X-Y| \le \mathbb{E}|X-\tfrac{1}{2}| + \mathbb{E}|Y-\tfrac{1}{2}| \le \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

(5)

I will explain the solution for $n=2$; the general case is similar.

So, let X, Y be random variables such that

$$\mathbb{P}\big((X,Y)\in D\big) = \int_D f(x)\, g(y)\, dx\,dy \quad \text{for } D \subseteq \mathbb{R}^2.$$

Use this for $D = A \times B$ $\Rightarrow$ by Fubini

$$(*) \qquad \mathbb{P}(X\in A,\ Y\in B) = \int_A f(x)\,dx \cdot \int_B g(y)\,dy \quad \text{for } A, B \subseteq \mathbb{R}.$$

As we are told, f and g may not be densities, that is $a := \int_{\mathbb{R}} f \neq 1$, $b := \int_{\mathbb{R}} g \neq 1$, but nevertheless for $A = B = \mathbb{R}$ we get

$$1 = \mathbb{P}(X\in\mathbb{R},\ Y\in\mathbb{R}) = ab.$$

For $B = \mathbb{R}$, we get from $(*)$:

$$\mathbb{P}(X\in A) = \int_A f(x)\,dx \cdot b\,;$$

and for $A = \mathbb{R}$, we get

$$\mathbb{P}(Y\in B) = \int_B g(y)\,dy \cdot a.$$

Multiplying these two identities, and using $ab=1$, we get

$$\mathbb{P}(X\in A)\,\mathbb{P}(Y\in B) = \int_A f(x)\,dx \cdot \int_B g(y)\,dy.$$

Together with $(*)$ this completes the proof.

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