Homework 5

Probability Theory (MATH 235A, Fall 2007)

1. Random series. Let $X_1, X_2, \dots$ be non-negative random variables. Prove the equality

$$\mathbb{E}\Big(\sum_{n=1}^{\infty} X_n\Big) = \sum_{n=1}^{\infty} \mathbb{E} X_n.$$

Explain in what sense the equality holds.

2. Sharpness of Chebyshev's inequality. For every $t \geq 1$, construct a random variable X with mean μ and variance σ^2, and such that Chebyshev's inequality becomes an equality:

$$\mathbb{P}(|X - \mu| \geq t\sigma) = \frac{1}{t^2}.$$

3. Averaging a density Prove that, for any random variable X and $a > 0$, one has

$$\int_{\mathbb{R}} \mathbb{P}(x < X < x + a)\, dx = a.$$

4. Uniform distribution Let X and Y be uniformly distributed random variables on $[0, 1]$. Show that, whatever the dependence between X and Y, one has

$$\mathbb{E}|X - Y| \leq \frac{1}{2}.$$

5. Independence Let $X_1, \dots, X_n$ be random variables such that

$$\mathbb{P}\big((X_1, \dots, X_n) \in A\big) = \int_A f(x)\, dx$$

for every Borel set A in $\mathbb{R}^n$. Assume that $f : \mathbb{R}^n \to \mathbb{R}$ can be factored as

$$f(x_1, \dots, x_n) = f_1(x_1) \cdots f_n(x_n)$$

for some non-negative measurable functions $f_k; \mathbb{R} \to \mathbb{R}$. Prove that $X_1, \dots, X_n$ are independent. Note that f_k are not assumed to be density functions. (However, if you feel a need of continuity, you may assume that f_k are continuous).