Homework 6. Solutions

①

(a) $\{X+Y=n\} = \bigcup_{m\in\mathbb{Z}} \{X=m,\ Y=n-m\}$.

The events in this union are disjoint. Therefore

$$\mathbb{P}(X+Y=n) = \sum_m \mathbb{P}(X=m,\ Y=n-m)$$

$$= \sum_m \mathbb{P}(X=m)\,\mathbb{P}(Y=n-m) \quad \text{by independence.}$$

QED.

(b) $\mathbb{P}(X=m) = \begin{cases} e^{-\lambda}\dfrac{\lambda^m}{m!}, & \text{for } m\geq 0 \\ 0, & \text{for } m<0 \end{cases}$ $\Bigg|$ $\mathbb{P}(Y=n-m) = \begin{cases} e^{-\mu}\dfrac{\mu^{n-m}}{(n-m)!} & \text{for } n-m\geq 0 \\ 0 & \text{for } n-m<0 \end{cases}$

By part (a),

$$\mathbb{P}(X+Y=n) = \sum_{m=0}^{n} e^{-\lambda}\frac{\lambda^m}{m!}\,e^{-\mu}\frac{\mu^{n-m}}{(n-m)!}$$

$$= e^{-(\lambda+m)} \sum_{m=0}^{n} \frac{\lambda^m \mu^{n-m}}{m!\,(n-m)!}$$

$$= e^{-(\lambda+m)} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} \lambda^m \mu^{n-m}$$

$$= e^{-(\lambda+n)} \frac{(\lambda+\mu)^n}{n!}.$$

QED.

(c) X = number of successes in n independent trials (probability of success in each trial $=p$).

Y = number of successes in m independent trials, which are also independent from the trials in X.

Hence $X+Y$ = number of successes in the total of $(n+m)$ independent trials. Hence Bernoulli with parameters $(n+m, p)$.

~~We need $\mathbb{E}X=0$, $\mathbb{E}$~~

We will find X, Y such that $\mathbb{E}X=0$, $\mathbb{E}Y=0$, $\mathbb{E}XY=0$ (therefore X, Y uncorrelated), but not independent.

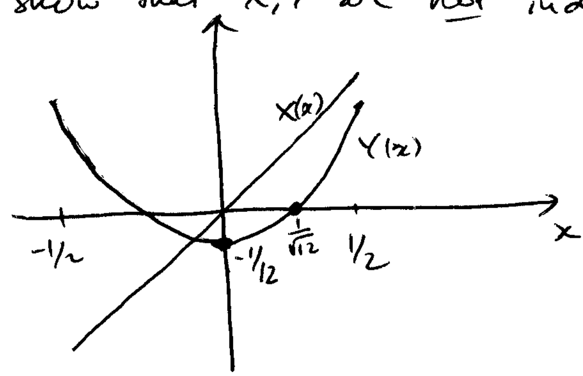
Let $X(x)=x$, $Y(x)=ax^2+b$ on $[-1/2, 1/2]$.

Then $\mathbb{E}X = \int_{-1/2}^{1/2} x\,dx = 0$

$$\mathbb{E}Y = \int_{-1/2}^{1/2} (ax^2+b)\,dx = \frac{a}{12}+b$$

$$\mathbb{E}XY = \int_{-1/2}^{1/2} x\,(ax^2+b)\,dx = 0 \qquad \text{(odd function)}$$

So if we choose $a=1$, $b=-1/12$, then X, Y are uncorrelated.
Now we show that X, Y are not independent.



$Y(x) > 0$ for all $x > \frac{1}{\sqrt{12}}$.

Hence $X > \frac{1}{\sqrt{12}}$ implies $Y > 0$, thus

$$\mathbb{P}\left(Y > 0,\ X > \tfrac{1}{\sqrt{12}}\right) = \mathbb{P}\left(X > \tfrac{1}{\sqrt{12}}\right). \qquad (*)$$

On the other hand, $\mathbb{P}(Y>0) < 1$. $\qquad (**)$

$(*)$ and $(**)$ contradict the independence of X, Y.

QED

(a) Let $H(z)$ be the distribution function of XY. Then

$$H(z) = \mathbb{P}(XY \leq z) = \int_{\mathbb{R}^2} \mathbb{1}_{\{xy \leq z\}} \, dP_{(X,Y)}$$

where $P_{(X,Y)}$ is the joint distribution of X, Y (i.e. the distribution of the random vector (X,Y)).

By the independence, $P_{(X,Y)}$ is the product measure of the distributions of P_X of X and P_Y of Y. So by Fubini,

$$H(z) = \int_{\mathbb{R}} \underbrace{\left(\int_{\mathbb{R}} \mathbb{1}_{\{xy \leq z\}} \, dP_X \right)}_{= \int_{\mathbb{R}} \mathbb{1}_{\{x \leq z/y\}} \, dP_X = P(X \leq z/y) = F(z/y)} dP_Y$$

Therefore,

$$\boxed{H(z) = \int_{\mathbb{R}} F(z/y) \, dG(y)}$$

(b) Let $H(z)$ be the distribution function of XY. Since $0 \leq X \leq 1$ and $0 \leq Y \leq 1$, $H(z) = 1$ if $z \geq 1$ and $H(z) = 0$ if $z \leq 0$.

Assume then $z \in (0,1)$. Then

$$H(z) = \int_0^1 F(z/y) \, dy$$

because $H(z) = \int_{\mathbb{R}} F(z/y) g(y) \, dy$ and the density $g(y)$ of Y is $\mathbb{1}_{[0,1]}$.

Now, $F(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x \geq 1 \end{cases}$

Hence $H(z) = \int_0^z dy + \int_z^1 \frac{z}{y} \, dy = z - z\ln z$.

Answer: $\boxed{H(z) = \begin{cases} 0, & z \leq 0 \\ z - z\ln z, & z \in (0,1) \\ 1, & z \geq 1 \end{cases}}$

Example: consider ~~a~~ random variables X_n with the following distribution:

$$P(X_n = n) = \frac{1}{\sqrt{n}} \; ; \qquad P(X_n = 0) = 1 - \frac{1}{\sqrt{n}}$$

Then $$\mathbb{E}X_n = n \cdot \frac{1}{\sqrt{n}} = \sqrt{n} \to \infty$$

therefore $X_n \not\to 0$ in L^1.

However, $$P(X_n \neq 0) = \frac{1}{\sqrt{n}} \to 0$$

Hence $X_n \to 0$ in probability.

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(a), (b): The unifying framework for both problems is a sequence of independent trials, ~~with probabilit~~ each successful with probability p $\left(p = \frac{1}{2} \text{ in (a)}, \; p = \frac{1}{6} \text{ in (b)}\right)$

We are interested in the expectation of the random variable

$X =$ number of trials until first success

Then $$P(X = k) = \underbrace{(1-p)^{k-1}}_{(k-1) \text{ failures}} \underbrace{p}_{k\text{'th trial} = \text{success}}, \qquad k = 1, 2, \ldots$$

Hence $$\mathbb{E}X = \sum_{k=1}^{\infty} k\,(1-p)^{k-1} p = \sum_{k=1}^{\infty} k\left[(1-p)^{k-1} - (1-p)^k\right]$$

$$= \sum_{k=1}^{\infty} k(1-p)^{k-1} - \sum_{k=1}^{\infty} k(1-p)^k = \sum_{k=0}^{\infty} (k+1)(1-p)^k - \sum_{k=0}^{\infty} k(1-p)^k$$

$$= \sum_{k=0}^{\infty} (1-p)^k = \frac{1}{p}.$$

Answer in (a): $\boxed{2 \text{ times}}$. Answer in (b): $\boxed{6 \text{ times}}$

(c) Let's call the distribution of X in the previous part "First success distribution with parameter p".

Let X_k, $k=1,\dots 6$, be the number of throws until the next new face appears. So the total number of ~~times~~ throws until all faces appear is $X_1 + \dots + X_6$.

(Note: X_k are not independent).

Clearly, $X_1 = 1$.

X_2 has the first success distribution with parameter $\frac{5}{6}$

X_3 " ——— " $4/6$

X_4 " ——— " $3/6$

X_5 " ——— " $2/6$

X_6 " ——— " $1/6$.

By parts (a), (b),

$$E(X_1 + \dots + X_6) = EX_1 + \dots + EX_6 = \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} = 14.7$$

(d) Similar with n instead of 6 $\Rightarrow$

$$E(X_1 + \dots + X_n) = n \sum_{k=1}^{n} \frac{1}{k} \sim \boxed{Cn \log n}$$