Midterm Exam. Solutions.

①

$$\mathbb{E}X = \int_0^\infty \mathbb{P}(X \geq x)\,dx = \sum_{n=1}^{\infty} \int_{n-1}^{n} \mathbb{P}(X \geq x)\,dx$$

For every $x \in (n-1, n]$, $\mathbb{P}(X \geq x) = \mathbb{P}(X \geq n)$

Hence $\int_{n-1}^{n} \mathbb{P}(X \geq x)\,dx = \mathbb{P}(X \geq n)$,

and therefore $\mathbb{E}X = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n)$.

②

The subset of <u>chosen</u> books is uniformly distributed in the set of all subsets on $\{1, \dots n\}$. Therefore, the set of <u>not chosen</u> books is also uniformly distributed in that set.

Let X denote the number of chosen books and Y denote the number of not chosen books.

By the property above, $X = Y$ in distribution.

Hence $\mathbb{E}X = \mathbb{E}Y$. Since $X + Y = n$, we have

$\mathbb{E}X + \mathbb{E}Y = n$

Therefore, $\mathbb{E}X = \mathbb{E}Y = n/2$.

QED

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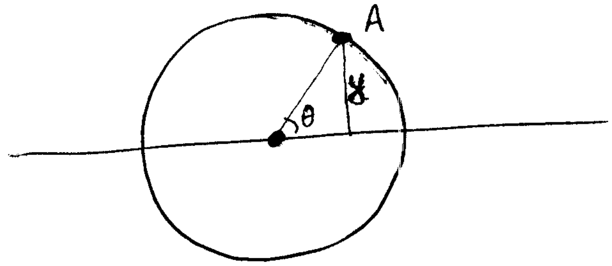
Because of the symmetry of the problem, we can assume that the line is the x-axis



The point A is uniform on the circle.

Let θ be the angle between the x-axis and A.

Then θ is uniform in $[0, 2\pi]$.

The distance in question is

$$y = |\sin\theta|.$$

The expectation of y is then

$$\frac{1}{2\pi}\int_0^{2\pi}|\sin\theta|\,d\theta = \frac{1}{\pi}\int_0^{\pi}\sin\theta\,d\theta = \frac{2}{\pi}.$$

QED.

(b) Since $0 \le X_n^2 \le n X_n \le n X$,

the monotonicity property of the expectation gives

$$\mathbb{E}(X_n^2) \le n\, \mathbb{E} X$$

Therefore the terms of the series are bounded as

$$n^{-p}\, \mathbb{E}(X_n^2) \le n^{1-p} \cdot \mathbb{E} X$$

Hence for $p > 2$, the series of these terms converges. QED.

(c) To prove the result for $p = 2$, we will have to be more accurate than in (b) when we bound X_n^2.

To this end, consider the random variables

$$Y_n := X \cdot \mathbb{1}_{\{n-1 \le X < n\}}, \qquad n = 1, 2, \ldots$$

Then $$X_n^2 = \sum_{k=1}^{n} Y_k^2 + n^2\, \mathbb{1}_{\{X > n\}}$$

Since $0 \le Y_k \le k$, we have $Y_k^2 \le k\, Y_k$. Therefore

$$X_n^2 \le \sum_{k=1}^{n} k\, Y_k + n^2 \mathbb{1}_{\{X > n\}}.$$

Then
$$\sum_{n=1}^{\infty} n^{-2}\, \mathbb{E}(X_n^2) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} n^{-2} k\, \mathbb{E} Y_k + \sum_{n=1}^{\infty} \mathbb{E}\, \mathbb{1}_{\{X > n\}} =: A + B.$$

Now, $$A = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} n^{-2} k\, \mathbb{E} Y_k$$ (the series can be rearranged because they have nonnegative terms)

Since $\sum_{n=k}^{\infty} n^{-2} \le C/k$ (where C is an absolute constant), we have

$$A \le C \sum_{k=1}^{\infty} \mathbb{E} Y_k = C \cdot \mathbb{E}\left(\sum_{k=1}^{\infty} Y_k\right) = C \cdot \mathbb{E} X$$

where the identity here follows by the Monotone Convergence Theorem (used for the partial sums $\sum_{k=1}^{N} Y_k$).

Furthermore, $B = \sum_{n=1}^{\infty} \mathbb{P}(X > n) \le \int_0^{\infty} \mathbb{P}(X > t)\, dt = \mathbb{E} X$. Hence $A + B \le (C+1)\, \mathbb{E} X$.

QED.