Midterm Exam

Probability Theory (MATH 235A, Fall 2007)

1. (10 pts) Let X be a non-negative random variable, which takes values $0, 1, 2, \ldots$ Prove that

$$\mathbb{E}X = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n).$$

2. (15 pts) From a set of n books, a subset is chosen uniformly at random. All subsets, including the empty set and the set of all books, are equally likely to be chosen. How many books are chosen on average?

3. (15 pts) A standard Cauchy random variable has density

$$f(x) = \frac{1}{\pi(1+x^2)}, \qquad -\infty < x < \infty.$$

Show that if X is a standard Cauchy random variable then $1/X$ is also a standard Cauchy random variable.

4. (20 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative function that satisfies

$$\int_{\mathbb{R}} f(x)\, dx = 1.$$

Prove that f is a density of some random variable X. (Construct X).

5. (15 pts) Let A be a point chosen uniformly at random from the circle $x^2 + y^2 = 1$. Compute the expectation of the distance from A to some fixed line through the origin.

6. Let X be a non-negative random variable such that $\mathbb{E}X$ exists.

(a) **(10 pts)** Show by example that $\mathbb{E}(X^2)$ may not exist.

(b) **(15 pts)** Consider the truncation $X_n = \min(X, n)$. Prove that for every $p > 2$,

$$\sum_{n=1}^{\infty} n^{-p}\, \mathbb{E}(X_n^2) < \infty.$$

(c) **(15 pts: bonus problem)**. Prove this for $p = 2$.