\[ \text{Var}(X-Y) = E \left[ (X-Y)^2 \right] - [E(X-Y)]^2. \]

\[ E(X-Y)^2 = E(X-Y)(X-Y) = E \left( X(X-Y) - Y(X-Y) \right) \]

\[ = E(X \cdot E(X-Y) - EY \cdot E(X-Y) \quad \text{by independence} \]

\[ = [E(X-Y)]^2. \]

Hence \( \text{var}(X-Y) = 0. \) Since \( \text{var}(X-Y) = E \left( (X-Y) - E(X-Y) \right)^2 = 0, \)

it follows that \( X-Y = E(X-Y) \) a.s. QED.

\[ \text{Bonus} \quad \text{By independence, since for } Z = X-Y \text{ we have} \]

\[ x = (x-y) + y \Rightarrow \psi_x(t) = \psi_z(t) \psi_y(t) \]

\[ x = z + y \Rightarrow \psi_y(t) = \psi_y(t) \psi_x(t) \quad (\ast) \]

\[ y = x - (x-y) \Rightarrow \psi_y(t) = \psi_x(t) \psi_z(t) \quad (\ast \ast) \]

By substitution of (\ast) into (\ast \ast) \( (\ast) \quad \)

\[ \psi_x(t) = \psi_z(t) \psi_x(t) \psi_z(t) \]

\[ \Rightarrow \psi_x(t) \left( 1 - |\psi_z(t)|^2 \right) = 0. \quad (\ast \ast \ast) \]

Since \( \psi_x(0) = 0 \) and \( \psi_x \) is a continuous function, \( \psi_x(t) \to 0 \) in some neighborhood of 0. Hence by (\ast \ast \ast), \( |\psi_z(t)| = 1 \) for \( t \) in some neighborhood of 0.
Now we claim that $Z = \text{const} \ a.s.$

Consider an independent copy $Z'$ of $Z$. We have

$$
\left(\begin{array}{c}
\Phi_{Z-Z'}(t) = \Phi_Z(t) \Phi_{Z'}(t) = 1 \\
\end{array}\right)
$$

in some neighborhood of $0$

Hence, since $\Phi_{Z-Z'}(t) = E \exp\left[i g(t-Z-Z')\right]$, it follows that

- by taking the real part that
  $$
  E\left(1 - \cos g(t-Z-Z')\right) = 0
  $$

  \(\Rightarrow\)

  $$
  1 - \cos g(t-Z-Z') = 0 \ a.s.
  $$

  \(\Rightarrow\)

  $$
  t(Z-Z') \in 2\pi \ Z \ a.s.
  $$

  \(\Rightarrow\)

  $$
  Z-Z' = 0 \ a.s. \text{ (because } t \text{ is arbitrary)}.
  $$

We now claim that $Z = \text{const} \ a.s.$ Indeed,

$$
1 = \Phi_{Z-Z'}(t) = \Phi_Z(t)^2 \quad \text{by (1)}, \quad \text{now fix all } t.
$$

Hence, $Z = \text{const} \ a.s.$

\[\text{It is now an exercise to show that } Z = 0 \ a.s.\]

\[\text{--- ALTERNATIVE PROOF OF BONUS ---} \]

\[\text{--- Only assuming integrability:} \]

by independence, $E(X|X-Y) = E(X)$; $E(Y|X-Y) = E(Y)$.

\(\Rightarrow\) $E(X-Y|X-Y) = E(X) - E(Y)$ by linearity.

\(\Rightarrow\) $X-Y = E(X-Y) = \text{const} \ a.s.$ \hspace{1cm} \(\text{Q.E.D.}\)
Let $F_n$ be the distribution function of $\frac{1}{n} \max X_k$.

$$F_n(x) = P\left(\frac{1}{n} \max X_k \leq x\right) = P(X_1 \leq nx)^n$$

$$= \left(1 - \int_{nx}^{\infty} \frac{dy}{\pi(1+y^2)}\right)^n.$$

If $x < 0$, then since $P(X_1 \leq nx) \to 0$ as $n \to \infty$, $F_n(x) \to 0$.

Suppose $x \geq 0$. Then

$$F_n(x) \sim \exp\left(-n \int_{nx}^{\infty} \frac{dy}{\pi(1+y^2)}\right) \quad \text{as} \quad n \to \infty,$$

$$= \exp\left(-\int_{\infty}^{\infty} \frac{dz}{\pi(1+n^2 z^2)}\right).$$

Thus, for $x > 0$

$$\lim_{x \to \infty} F_n(x) = \exp\left(-\int_{\infty}^{\infty} \frac{dz}{\pi(1+z^2)}\right) = \exp\left(-\frac{1}{\pi x}\right) = F(x),$$

where $F(x)$ is the d.f. of an exponential r.v. with parameter

$$\lambda = \frac{1}{\bar{X}}.$$
Let $X_n = i.i.d. \ N(0,1)$ r.v.'s,

$Y_n = B_n \cdot Z_n$, where $B_n, Z_n$ indep. r.v.'s such that

- $Z_n$ = indep. Cauchy r.v.'s
- $B_n = (\text{Bernoulli with parameter } \frac{1}{n^2}) = \begin{cases} 1, \text{ prob } \frac{1}{n^2} \\ 0, \text{ otherwise} \end{cases}$

Then $Y_n$ has no variance (by conditioning on $\mathbb{E}B_n = 1$)

and

Let $S'_n = X_1 + \cdots + X_n$, $S''_n = Y_1 + \cdots + Y_n$.

$P(S''_n \neq 0) \leq P(\exists k \leq 1: B_k = 1) \leq \sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$

Hence $\frac{S''_n}{\sqrt{n}} \rightarrow 0$ in probability. \hspace{1cm} (*)

Also by the CLT (or just by the property of $N(0,1)$),

$\frac{S'_n}{\sqrt{n}} \rightarrow N(0,1)$ in distribution. \hspace{1cm} (**)

Problem 3 of Midterm shows that $(*) + (**)$ \Rightarrow

$\frac{S_n}{\sqrt{n}} = \frac{S'_n}{\sqrt{n}} + \frac{S''_n}{\sqrt{n}} \rightarrow N(0,1)$ in distribution.

So $X_n = U_n + Y_n$ completes the proof. \hspace{1cm} Q.E.D.
\[ EX_N = \sum_{k=1}^{M} \left( \sum_{N=k+1}^{\infty} X_k \, dP \right) \]

\[ = \sum_{k=1}^{M} \left( \sum_{N=k+1}^{\infty} X_k \, dP - \sum_{N=k+1}^{\infty} X_k \, dP \right) \, \text{Note that } \{N=k+1\} \in F_{k+1} \]

\[ \text{(N is stopping time!)} \]

Hence, by the martingale definition and the conditional expectation,

\[ \sum_{N=k+1}^{\infty} X_k \, dP = \sum_{N=k}^{\infty} X_{k-1} \, dP = : a_k \]

Then the sum above transforms into

\[ EX_N = \sum_{k=1}^{M} (a_k - a_{k+1}) = a_1 - a_{M+1} \]

\[ = a_1 \quad \text{because } a_{M+1} = 0 \text{ since } P(N_{M+1} = 0) = 0 \]

\[ = \sum_{N=1}^{\infty} X_1 = \mathbb{E}X_1 \]

Q.E.D.