FINAL EXAM Solutions

1

$$\mathrm{Var}(X-Y) = \mathbb{E}\left[(X-Y)^2\right] - \left[\mathbb{E}(X-Y)\right]^2.$$

$$\mathbb{E}(X-Y)^2 = \mathbb{E}(X-Y)(X-Y) = \mathbb{E}\left(X(X-Y) - Y(X-Y)\right)$$
$$= \mathbb{E}X \cdot \mathbb{E}(X-Y) - \mathbb{E}Y \cdot \mathbb{E}(X-Y) \quad \text{by independence}$$
$$= \left[\mathbb{E}(X-Y)\right]^2.$$

Hence $\mathrm{var}(X-Y) = 0$. Since $\mathrm{var}(X-Y) = \mathbb{E}\left((X-Y) - \mathbb{E}(X-Y)\right)^2 = 0$,

it follows that $X - Y = \mathbb{E}(X-Y)$ a.s. QED.

(without even integrability of X, Y!)

Bonus By independence ~~since~~ for $Z = X - Y$ we have

~~$X = (X-Y)+Y \Rightarrow \varphi_X(t) = \varphi_Z(t)\varphi$~~

$$X = Z + Y \Rightarrow \qquad \varphi_X(t) = \varphi_Z(t)\,\varphi_Y(t) \qquad (*)$$

$$Y = X - (X-Y) \Rightarrow \qquad \varphi_Y(t) = \varphi_X(t)\,\overline{\varphi_Z(t)}. \qquad (**)$$

By substitution of $(**)$ into $(*)$

$$\varphi_X(t) = \varphi_Z(t)\,\varphi_X(t)\,\overline{\varphi_Z(t)}$$

$$\Rightarrow \varphi_X(t)\left(1 - |\varphi_Z(t)|^2\right) = 0. \qquad (***)$$

Since $\varphi_X(0)=0$ and φ_X is a continuous function, $\varphi_X(t) \neq 0$ in some neighborhood of 0. Hence by $(***)$,

$$|\varphi_Z(t)| = 1 \quad \text{for } t \text{ in some neighborhood of } 0.$$

Now we claim that $Z = \text{const}$ a.s.

Consider an independent copy Z' of Z. We have

$$(\star) \quad \varphi_{Z-Z'}(t) = \varphi_Z(t)\,\overline{\varphi_{Z'}(t)}$$

$$= \varphi_Z(t)\,\overline{\varphi_Z(t)} = |\varphi_Z(t)|^2 = 1 \quad \text{in some neighborhood of } 0$$

~~Hence~~ Since $\varphi_{Z-Z'}(t) = \mathbb{E}\exp[it(Z-Z')]$, it follows ~~that~~ by taking the real part that

$$\mathbb{E}\left(1 - \cos t(Z-Z')\right) = 0$$

$$\Rightarrow 1 - \cos t(Z-Z') = 0 \quad \text{a.s.}$$

$$\Rightarrow t(Z-Z') \in 2\pi\mathbb{Z} \quad \text{a.s.}$$

This repeats Problem 3 of HW 3

$$\Rightarrow Z - Z' = 0 \quad \text{a.s.} \quad (\text{because } t \text{ is arbitrary}).$$

We now claim that $Z = \text{const}$ a.s. Indeed,

$$1 = \varphi_{Z-Z'}(t) = |\varphi_Z(t)|^2 \quad \text{by } (\star), \text{ now for } \underline{\underline{\text{all}}}\ t.$$

$$\Rightarrow |\varphi_Z(t)| = 1 \quad \text{a.s.}$$

~~Hence~~ Z

It is now an exercise to show that $Z = 0$ a.s.

ALTERNATIVE PROOF OF BONUS, ONLY assuming integrability:

By independence, $\mathbb{E}(X \mid X-Y) = \mathbb{E}(X)$; $\mathbb{E}(Y \mid X-Y) = \mathbb{E}(Y)$.

$$\Rightarrow \underbrace{\mathbb{E}(X-Y \mid X-Y)}_{= X-Y} = \mathbb{E}(X \mid X-Y) - \mathbb{E}(Y \mid X-Y) \quad \text{by linearity}$$

$$= \mathbb{E}(X) - \mathbb{E}(Y).$$

$$\Rightarrow X - Y = \mathbb{E}(X-Y) = \text{const}. \qquad \text{QED.}$$

2

Let F_n be the distribution function of $\frac{1}{n} \max_{k \le n} X_k$.

$$F_n(x) = P\left(\frac{1}{n} \max_{k \le n} X_k \le x \right) = P(X_1 \le nx)^n$$

$$= \left(1 - \int_{nx}^{\infty} \frac{dy}{\pi(1+y^2)} \right)^n .$$

If $x < 0$, then since $P(X_1 \le nx) \to 0$ as $n \to \infty$, $F_n(x) \to 0$.

Suppose $x > 0$. Then

$$F_n(x) \sim \exp\left(-n \int_{nx}^{\infty} \frac{dy}{\pi(1+y^2)} \right) \quad \text{as } n \to \infty$$

$$= \exp\left(- \int_{x}^{\infty} \frac{n^2 dz}{1+n^2z^2} \right)$$

Thus, ~~for x→~~

$$\lim_{x \to \infty} F_n(x) = \exp\left(- \int_{x}^{\infty} \frac{dz}{\pi z^2} \right) = \exp\left(-\frac{1}{\pi x} \right) = F(x)$$

where $F(x)$ is the d.f. of an exponential r.v. with parameter

$$\lambda = \frac{1}{\pi}.$$

~~3~~

~~Let X_n = i.i.d. $N(0,1)$ r.v.'s,~~
~~$Y_n = \frac{1}{n} \cdot Z_n$, where Z_n = i.i.d. Cauchy r.v.'s.~~
~~By Problem 4 of HW 2 and Problem 5 of HW 1,~~
~~$Y_n \to 0$ in probability~~

(3)

Let U_n = i.i.d. $N(0,1)$ r.v's,

$Y_n = B_n \cdot Z_n$, where B_n, Z_n indep. r.v's such that

Z_n = indep. Cauchy r.v's

$B_n = \left(\text{Bernoulli with parameter } \frac{1}{n^2}\right) = \begin{cases} 1, & \text{prob } 1/n^2 \\ 0, & \end{cases}$

Then Y_n has no variance (by conditioning on $\{B_n = 1\}$)

Let $S_n' = U_1 + \dots + U_n$; $S_n'' = Y_1 + \dots + Y_n$.

$$P(S_n'' \neq 0) \leq P(\exists k \leq 1 : B_n = 1) \leq \sum_1^n \frac{1}{n^2} = \frac{1}{n} \to 0 \text{ as } n \to \infty$$

Hence $\frac{S_n''}{\sqrt{n}} \to 0$ in probability. (*)

By the CLT (or just by the property of $N(0,1)$),

$\frac{S_n'}{\sqrt{n}} \to N(0,1)$ in distribution. (**)

Problem 3 of Midterm shows that (*) + (**) $\Rightarrow$

$$\frac{S_n}{\sqrt{n}} = \frac{S_n'}{\sqrt{n}} + \frac{S_n''}{\sqrt{n}} \to N(0,1) \text{ in distribution.}$$

So $X_n := U_n + Y_n$ completes the proof. Q.E.D.

(5) Let $M =$ bound on the stopping time.

$$\mathbb{E}X_N = \sum_{k=1}^{M} \int_{\{N=k\}} X_k \, dP$$

$$= \sum_{k=1}^{M} \left(\int_{\{N\geq k\}} X_k \, dP - \int_{\{N\geq k+1\}} X_k \, dP \right).$$ Note that $\{N \geq k\} \in \mathcal{F}_{k-1}$ (N is stopping time!)

~~Now~~ Hence, by the martingale definition and the conditional expectation,

$$\int_{\{N\geq k\}} X_k \, dP = \int_{\{N\geq k\}} X_{k-1} \, dP =: a_k$$

Then the sum above transforms into

$$\mathbb{E}X_N = \sum_{k=1}^{M} (a_k - a_{k+1}) = a_1 - a_{M+1}$$

$= a_1$ because $a_{M+1} = 0$ since $P(N \geq M+1) = 0$ ~~$N \leq M$ a.s.~~.

$$= \int_{\{N\geq 1\}} X_1 = \mathbb{E}X_1$$

QED.