Final Exam. Probability Theory (MATH 235B, Winter 2008)

Independent work. Please only use class or HW posted material.

1. Let X and Y be random variables with finite variances. Assume that $X-Y$ is independent of X, and $X-Y$ is independent of Y. Show that $X-Y$ is a.s. constant.

Bonus problem (full credit only): prove the same result for arbitrary random variables X and Y (without even integrability assumption).

(Hint for the bonus problem: Let ϕ be the characteristic function of X and ψ be the characteristic function of $X-Y$. First show that

$$\phi(x)(1-|\psi(x)|^2)=0 \qquad \textit{for every } x \in \mathbb{R}.$$

Conclude that $|\psi(x)|=1$ for all x in some neighborhood of zero. Deduce from this that $X-Y=0$ a.s.)

2. Let (X_n) be independent Cauchy random variables (i.e. density equals $1/\pi(1+x^2)$.) Show that

$$\frac{1}{n}\max_{k\le n} X_k \to \frac{1}{T} \qquad \text{in distribution,}$$

where T is an exponential random variable; determine the parameter λ of T.

3. Give an example of mean zero indepednent random variables (X_n) without finite variances, but which satisfy the conclusion of the Central Limit Theorem: for $S_n = X_1 + \cdots X_n$, one has

$$\frac{S_n}{\sqrt{n}} \to N(0,1) \qquad \text{in distribution,}$$

where $N(0,1)$ is a standard normal random variable.

(Hint: consider $X_n = Y_n + Z_n$, where Y_n are some nice random variables and Z_n are bad, but the contribution of Z_n wears off for large n).

4. Let $(X_n)_{n\ge 0}$ be a martingale, and $N \ge 1$ be a bounded stopping time. Show that

$$\mathbb{E}X_N = \mathbb{E}X_1.$$

(Hint: decompose the expectation into a sum of the integrals $\int_{N=k} X_k \, d\mathbb{P} = \int_{N\ge k} X_k \, d\mathbb{P} - \int_{N\ge k+1} X_k \, d\mathbb{P}$, and use the martingale properties and the definition of the conditional expectation).