
Independent work. Please only use class or HW posted material.

1. Let $X$ and $Y$ be random variables with finite variances. Assume that $X - Y$ is independent of $X$, and $X - Y$ is independent of $Y$. Show that $X - Y$ is a.s. constant.

**Bonus problem (full credit only):** prove the same result for arbitrary random variables $X$ and $Y$ (without even integrability assumption).

(Hint for the bonus problem: Let $\phi$ be the characteristic function of $X$ and $\psi$ be the characteristic function of $X - Y$. First show that
\[
\phi(x)(1 - |\psi(x)|^2) = 0 \quad \text{for every } x \in \mathbb{R}.
\]
Conclude that $|\psi(x)| = 1$ for all $x$ in some neighborhood of zero. Deduce from this that $X - Y = 0$ a.s.)

2. Let $(X_n)$ be independent Cauchy random variables (i.e. density equals $1/\pi(1 + x^2)$.) Show that
\[
\frac{1}{n} \max_{k \leq n} X_k \to \frac{1}{T} \quad \text{in distribution},
\]
where $T$ is an exponential random variable; determine the parameter $\lambda$ of $T$.

3. Give an example of mean zero indepednent random variables $(X_n)$ without finite variances, but which satisfy the conclusion of the Central Limit Theorem: for $S_n = X_1 + \cdots + X_n$, one has
\[
\frac{S_n}{\sqrt{n}} \to N(0, 1) \quad \text{in distribution},
\]
where $N(0, 1)$ is a standard normal random variable.

(Hint: consider $X_n = Y_n + Z_n$, where $Y_n$ are some nice random variables and $Z_n$ are bad, but the contribution of $Z_n$ wears off for large $n$).

4. Let $(X_n)_{n \geq 0}$ be a martingale, and $N \geq 1$ be a bounded stopping time. Show that
\[
\mathbb{E}X_N = \mathbb{E}X_1.
\]

(Hint: decompose the expectation into a sum of the integrals $\int_{N=k} X_k \, d\mathbb{P} = \int_{N \geq k} X_k \, d\mathbb{P} - \int_{N \geq k+1} X_k \, d\mathbb{P}$, and use the martingale properties and the definition of the conditional expectation).