1.

The number of heads $S_n$ is a binomial random variable with parameters $(n, \frac{1}{2})$, where $n = 14,400$.

By De Moivre–Laplace CLT,

$$P(S_n \leq 7,428) = P(S_n \leq \frac{n}{2} + 228)$$

$$= P\left( \frac{S_n - \frac{n}{2}}{\frac{n}{2}} \leq \frac{228}{60} = 3.8 \right)$$

$$\sim P(\gamma \leq 3.8) \quad \text{where} \quad \gamma \sim N(0,1)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{3.8} e^{-x^2/2} \, dx \approx 0.9999$$

2.

The birth of a girl will be referred to as a "success".

Then we have $n = 10,000$ independent trials, and the probability of a success at each trial is given by $p = 0.488$.

The binomial random variable $S_n$ with parameters $(n, p)$, which counts the number total number of girls, has mean $np = 4880$ and standard deviation $\sqrt{np(1-p)} = 50$. Then:

(a) $P(\overline{S_n} \geq 10,000 - S_n) = P(S_n > 5000) \approx P\left(\frac{S_n - np}{\sqrt{np(1-p)}} > 2.4\right)$

$\overline{S_n}$ girls, $n-S_n$ boys

$$\approx P(\gamma > 2.4) \quad \text{where} \quad \gamma \sim N(0,1)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{2.4}^{\infty} e^{-x^2/2} \, dx \approx 0.0082$$
\( (6) \)  
\[
\Pr \left( \frac{10,000 - S_n}{\frac{n-p}{\sqrt{np(1-p)}}} \leq 22/5 \right) = \Pr \left( g \leq 22/5 \right) \quad \text{where } g \sim N(0,1)
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{22/5} e^{-x^2/2} \, dx \approx 1.
\]

\( 3 \)  
[ Solutions skipped ]

\( 4 \)

(a) Consider the random variable \( X_n \) uniformly distributed on \( n \) values \( \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} \). Its distribution function \( F_n \) has this form:

\[
F_n(x) = \frac{\ln(n) x}{n}, \quad x \in [0,1].
\]

Clearly, \( F_n(x) \to F(x) = \begin{cases} 
0, & x < 0 \\
\frac{x}{1}, & x \in (0,1] \\
1, & x \geq 1 
\end{cases} \)

\( F(x) \) is the distribution function of the uniform random variable \( X \) on \([0,1]\). We have shown:

\( X_n \to X \) in distribution.
However, if $A = \mathbb{Q}$ (the set of rationals) then
\[
P(X_n \in A) = 1,
\]
while
\[
P(x \in A) = 0 \text{ because } A \text{ is countable.}
\]
Hence
\[
P(X_n \in A) \not\rightarrow P(x \in A).
\]
QED.

(b). Consider the subset $I \subset [0, 1]$ defined as
\[
I_i = \bigcup_{k=1}^\infty \left[ \frac{k}{n} - \frac{1}{2n}, \frac{k}{n} \right]
\]

Let $X_n$ be a random variable uniformly distributed on $I$.

Its distribution function $F_n$ has the form.

\[
y = F_n(x)
\]

Similarly to (a), $F_n(x) \rightarrow F(x)$, the distribution function of a uniform random variable $X$ on $[0, 1]$.

The density $f_n$ of $X_n$ has the form $f_n(x) = 2 \cdot 1_{I}(x)$

while the density $f$ of $X$ has the form $f(x) = 1_{[0,1]}(x)$

Therefore, $f_n(x) \not\rightarrow f(x)$ for any $x \in (0,1]$.

QED.
Since the convergence in probability always implies convergence in distribution, it remains to show that

\[ X_n \rightarrow a \text{ in distribution} \implies X_n \rightarrow a \text{ in probability}. \]

To this end, put \( \varepsilon := |x - a| \).

If \( x > a \) then \( P(X_n \leq x) \geq P(|X_n - a| < \varepsilon) \rightarrow 1 \) as \( n \rightarrow \infty \).

If \( x < a \) then \( P(X_n \leq x) \leq P(|X_n - a| > \varepsilon) \rightarrow 0 \) as \( n \rightarrow \infty \).

Hence \( P(X_n \leq x) \rightarrow \begin{cases} 1, & \text{for } x > a \\ 0, & \text{for } x < a \end{cases} = P(x \leq a) \)

for all \( x \neq a \). Since \( x = a \) is the point of discontinuity of the constant random variable \( a \), we have proved that

\[ X_n \rightarrow a \text{ in probability}. \]

QED.