Homework 1 Solutions.

(1)

The number of heads S_n is a Binomial random variable with parameters $(n, \frac{1}{2})$, where $n = 14,400$.

By De Moivre - Laplace CLT,

$$\mathbb{P}(S_n \leq 7,428) = \mathbb{P}(S_n \leq \tfrac{n}{2} + 228)$$

$$= \mathbb{P}\left(\frac{S_n - n/2}{\sqrt{n}/2} \leq \frac{228}{60} = 3.8 \right)$$

$$\sim \mathbb{P}(g \leq 3.8) \quad \text{where } g \text{ is } N(0,1)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{3.8} e^{-x^2/2}\, dx \approx \boxed{0.9999}$$

(2)

The birth of a girl will be referred to as a "success". Then we have $n = 10,000$ independent trials and the probability of a success at each trial is given by $p = 0.488$.

The Binomial random variable S_n with parameters (n, p), which counts the ~~nmb~~ total number of girls, has mean $np = 4880$ and standard deviation $\sqrt{np(1-p)} = 50$. Then:

(a) $$\mathbb{P}\{\underbrace{S_n}_{\text{girls}} \geq \underbrace{10,000 - S_n}_{\text{Boys}}\} = \mathbb{P}(S_n \geq 5000) \approx \mathbb{P}\left(\frac{S_n - np}{\sqrt{np(1-p)}} \geq 2.4 \right)$$

$$\approx \mathbb{P}(g \geq 2.4) \quad \text{where } g \text{ is } N(0,1)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{2.4}^{\infty} e^{-x^2/2}\, dx \approx \boxed{0.0082}$$

(b) $\mathbb{P}(\underbrace{10{,}000 - S_n}_{\text{boys}} \geq \underbrace{S_n}_{\text{girls}} + 200) = \mathbb{P}(S_n \leq 5{,}100)$

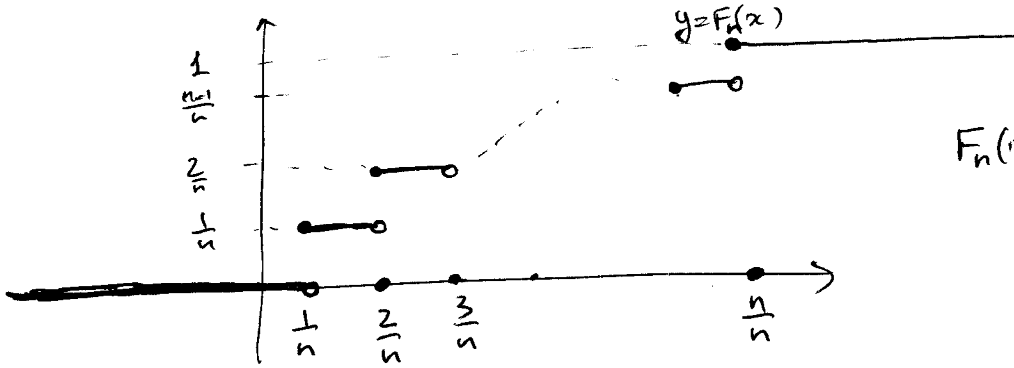
$$= \mathbb{P}\left(\frac{S_n - np}{\sqrt{np(1-p)}} \leq 22/5\right)$$

$$\simeq \mathbb{P}(g \leq 22/5) \quad \text{where } g \text{ is } N(0,1)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{22/5} e^{-x^2/2}\, dx \approx \boxed{1}.$$

3 [Solutions skipped].

4 (a). Consider the random variable X_n uniformly distributed on n values $\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}$. Its distribution function F_n has this form:



$$F_n(x) = \frac{\lfloor nx \rfloor}{n}, \quad x \in [0,1].$$

Clearly, $F_n(x) \to F(x) = \begin{cases} x, & x \in [0,1], \\ 0, & x < 0 \\ 1, & x > 1 \end{cases}$

$F(x)$ is the distribution function of the uniform random variable X on $[0,1]$. We have shown:

$$X_n \to X \quad \text{in distribution}.$$

However, if $A = \mathbb{Q}$ (the set of rationals) then

$$\mathbb{P}(X_n \in A) = 1,$$

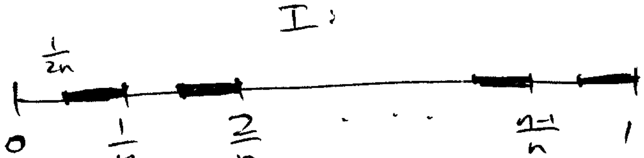
while $\quad \mathbb{P}(X \in A) = 0 \quad$ because A is countable.

Hence $\quad \mathbb{P}(X_n \in A) \not\to \mathbb{P}(X \in A)$. $\quad$ QED.

(b). Consider the subset $I \subset [0,1]$ defined as

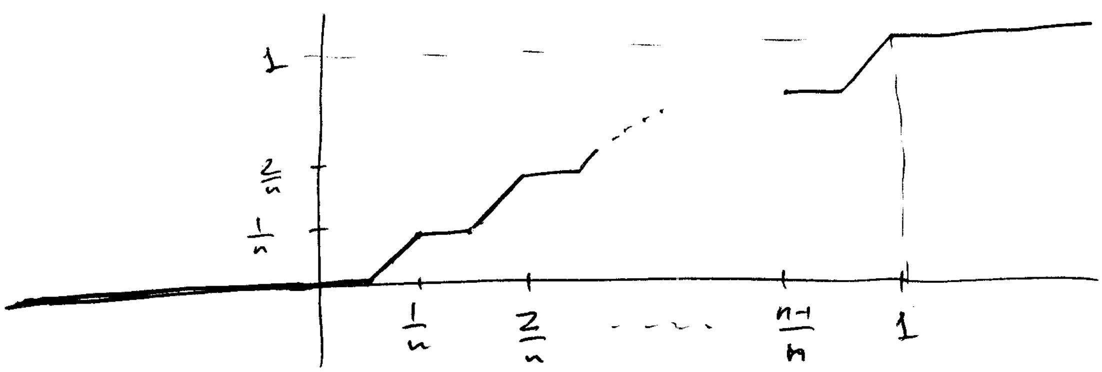
$$I := \bigcup_{k=1}^{n} \left[\frac{k}{n} - \frac{1}{2n}, \frac{k}{n}\right]$$

I :



Let X_n be a random variable uniformly distributed on I.

Its distribution function F_n has this form:



$y = F_n(x)$

Similarly to (a), $F_n(x) \to F(x)$, the distribution function of a uniform random variable X on $[0,1]$.

The density f_n of X_n has the form $\quad f_n(x) = 2 \cdot \mathbb{1}_I(x)$

while the density f of X has the form $\quad f(x) = \mathbb{1}_{[0,1]}(x)$.

Therefore, $\quad f_n(x) \not\to f(x) \quad$ for <u>any</u> $x \in [0,1]$.

QED.

(5).

Since the convergence in probability always implies convergence in distribution, it remains to show that

$$X_n \to a \text{ in distribution } \implies X_n \to a \text{ in probability.}$$

To this end, put $\varepsilon := |x-a|$.

If $x > a$ then $\mathbb{P}(X_n \le x) \geq \mathbb{P}(|X_n - a| < \varepsilon) \to 1$ as $n \to \infty$

If $x < a$ then $\mathbb{P}(X_n \le x) \le \mathbb{P}(|X_n - a| \geq \varepsilon) \to 0$ as $n \to \infty$.

Hence $\mathbb{P}(X_n \le x) \to \left\{ \begin{matrix} 1, \text{ for } x > a \\ 0, \text{ for } x < a \end{matrix} \right\} = \mathbb{P}(a \le x)$

for all $x \neq a$. Since $x = a$ is the point of discontinuity of the constant random variable a, we have proved that

$$X_n \to a \text{ in probability.}$$

QED.