Homework 1

Probability Theory (MATH 235B, Winter 2008)

1. A fair coin is tossed 14,400 times. What is the probability that a head occurs at most 7,428 times?

You may use a computer software to evaluate the standard probability distributions (like normal, Poisson, etc.), or look up a table of their values.

2. The probability of a boy at birth is approximately 0.512. Evaluate the probability that among 10,000 newborn babies:

(a) there are at least as many girls as boys;

(b) there are at least 200 more boys than girls?

3. Let $S_n = X_1 + \cdots + X_n$ where X_k are independent random variables such that

$$\mathbb{P}(X_k = 1) = \mathbb{P}(X_k = -1) = \frac{1}{2}.$$

Prove that the random variable $S_n' = S_n/\sqrt{n}$ obeys the following asymptotics for every $\theta > 0$:

$$\mathbb{P}\Big(S_n' > x + \frac{\theta}{x} \;\Big|\; S_n' > x\Big) \to e^{-\theta} \qquad \text{as } x \to \infty.$$

(You may use Proposition 6.9 of the course notes.)

4. Let X and $X_1, X_2, \ldots$ be random variables.

(a) Show by example that the following statement is **not** true in general:

> If $X_n \to X$ in distribution then $\mathbb{P}(X_n \in A) \to \mathbb{P}(X \in A)$ for every Borel set A.

(Hint: consider the random variable X_n uniformly distributed on n values $1/n, 2/n, \ldots, n/n$. Identify X.)

(b) Assume that the random variables X_n have densities f_n, and the random variable X has density f. Show by example that the following statement is **not** true in general:

> If $X_n \to X$ in distribution then $f_n(x) \to f(x)$ for every $x \in \mathbb{R}$.

(Hint: modify the example in (a) by including small intervals around the values k/n.)

5. Let a be a real number and $X_1, X_2, \ldots$ be random variables. Prove that $X_n \to a$ in distribution if and only if $X_n \to a$ in probability. Here we regard a as a random variable that takes the value a with probability 1.