Homework 2 Solutions

①

Let $\varepsilon > 0$. Since $F(x) \to 1$ as $x \to \infty$ and $F(x) \to 0$ as $x \to -\infty$, we can find $M = M(\varepsilon) > 0$ such that

$$F(-M) \leq \varepsilon, \qquad F(M) \geq 1-\varepsilon.$$

By increasing M, we may assume that M is a point of continuity of F. So by convergence in distribution $\exists\, n_0$ s.t.

$$F_n(-M) \leq 2\varepsilon, \qquad F_n(M) \geq 1-2\varepsilon, \qquad n \geq n_0.$$

By the monotonicity of F and F_n, we have

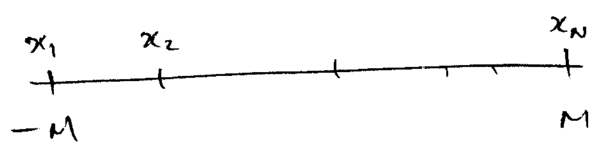
$$|F_n(x) - F(x)| \leq F_n(x) + F(x) \leq \varepsilon + 2\varepsilon = 3\varepsilon \quad \text{for } x \leq -M$$

and

$$|F_n(x) - F(x)| \leq |1-F_n(x)| + |1-F(x)| \leq \varepsilon + 2\varepsilon = 3\varepsilon \quad \text{for } x \geq M.$$

Now we consider $x \in [-M, M]$.
Since F is continuous on $[-M, M]$, it is uniformly continuous on $[-M, M]$. Therefore, we can partition the interval $[-M, M]$ into $N = N(M)$ sub-intervals with endpoints $(x_k)_{k=1}^{N}$ such that

$$F(x_k) - F(x_{k-1}) < \varepsilon, \qquad k = 1, \dots, N. \qquad (*)$$



Since $F_n(x_k) \to F(x_k)$ for every $k=1,\dots N$,

we can choose n_1 s.t.

$$|F_n(x_k) - F(x_k)| < \varepsilon \quad \text{for all } k=1,\dots N. \qquad (**)$$

let $x \in [-M,M]$. Then $\exists k: x \in [x_{k-1}, x_k]$.

Now, writing $a \overset{\varepsilon}{\approx} b$ for $|a-b| \le \varepsilon$, we have :

$$F(x) \overset{\varepsilon}{\approx} F(x_{k-1}) \qquad \text{since } \underset{F(x_{k-1})}{\le} F(x) \le F(x_k) \text{ and using } (*)$$

$$\overset{\varepsilon}{\approx} F_n(x_{k-1}) \qquad \text{by } (**)$$

$$\le F_n(x) \qquad \text{because } x_{k-1} \le x.$$

Similarly,

$$F(x) \overset{\varepsilon}{\approx} F(x_k)$$

$$\overset{\varepsilon}{\approx} F_n(x_k)$$

$$\ge F_n(x).$$

We have shown that

$$F_n(x) - 2\varepsilon \le F(x) \le F_n(x) + 2\varepsilon,$$

or equivalently

$$|F(x) - F_n(x)| \le 2\varepsilon \qquad \text{for all } x \in [-M,M].$$

This completes the proof.

2

1) Let $X_n \to X$ in distribution. Since $\mathbb{P}(X = k+\frac{1}{2}) = \mathbb{P}(k-\frac{1}{2}) = 0$ for all $k \in \mathbb{Z}$,

the definition of the convergence ~~of~~ in distribution yields

$$\mathbb{P}(X_n = k) = \mathbb{P}(k-\tfrac{1}{2} < X_n \le k+\tfrac{1}{2}) \qquad (\text{since } X_n \in \mathbb{Z})$$

$$\to \mathbb{P}(k-\tfrac{1}{2} < X \le k+\tfrac{1}{2}) \qquad \text{as } n \to \infty.$$

The proof would be complete if we show that $X \in \mathbb{Z}$ a.s., for in this case

$$\mathbb{P}(k-\tfrac{1}{2} \le X \le k+\tfrac{1}{2}) = \mathbb{P}(X=k),$$

as needed.

To show that $X \in \mathbb{Z}$ a.s. it suffices to prove for every $k \in \mathbb{Z}$ that

$$\mathbb{P}(k < X < k+1) = 0.$$

Now,

$$\mathbb{P}(k < X < k+1) = \mathbb{P}\left(k+\tfrac{1}{N} < X < k+1-\tfrac{1}{N} \text{ for some } N \in \mathbb{N}\right)$$

$$\le \sum_{N=1}^{\infty} \underbrace{\mathbb{P}(k+\tfrac{1}{N} < X < k+1-\tfrac{1}{N})}_{= \lim_n \mathbb{P}(k+\frac{1}{N} < X_n < k+1-\frac{1}{N}) = \lim_n 0 = 0} \; = 0.$$

Q.E.D.

2) The "only if" part is simple.

③

By Skorokhod's Representation Theorem, we can find

Y_n distributed as X_n,

Y distributed as X,

such that $Y_n \to Y$ pointwise (thus a.s)

Thus this limit Y is unique. ~~~~

~~~~ Since X is distributed identically with Y, the distribution of X is uniquely defined.

④

Let $\varepsilon, \delta > 0$. By ~~~~ properties of distribution functions, there exists x s.t.

$$P(|X| \geq x) \leq \delta. \quad \text{and} \quad P(|X| = x) = 0.$$

Further choose n_0 st.

$$|\delta_n| \leq \frac{\varepsilon}{x} \quad \text{for all } n \geq n_0,$$

and $$\left| P(|X_n| \geq x) - P(|X| \geq x) \right| < \delta \quad \text{for all } n \geq n_0.$$

(By convergence in distribution)

Then

$$P(|\delta_n X_n| \geq \varepsilon) \leq P(|X_n| \geq x) \leq P(|X| \geq x) + \delta \leq 2\delta.$$

for all $n \geq n_0$.

This shows that $\delta_n X_n \to 0$ in distribution.

⑤ Is a direct computation.
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