Homework 2
Probability Theory (MATH 235B, Winter 2008)

1. Consider random variables $X_1, X_2, \ldots$ with distribution functions $F_1, F_2, \ldots$. Suppose that $X_n \to X$ in distribution, and the distribution function $F$ of $X$ is continuous. Prove that $F_n \to F$ in the sup norm, i.e.

$$\|F_n - F\|_\infty := \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \to 0.$$ 

2. Consider integer valued random variables $X_1, X_2, \ldots$, and a random variable $X$. Show that $X_n \to X$ in distribution if and only if

$$\mathbb{P}(X_n = k) \to \mathbb{P}(X = k) \quad \text{for all } k \in \mathbb{Z}.$$ 

3. (Uniqueness Lemma) Suppose that random variables $X_n$ converge to some random variable $X$ in distribution. Show that the distribution of $X$ is uniquely defined. (Hint: you may use Skorokhod’s Representation Theorem.)

4. Suppose $X_n \to X$ in distribution, and $a_n$ are real numbers such that $a_n \to 0$ as $n \to \infty$. Show that $a_nX_n \to 0$ in distribution.

5. (Convergence of maxima) Let $X_1, X_2, \ldots$ be independent random variables, and let

$$M_n := \max_{k \leq n} X_k, \quad n = 1, 2, \ldots$$

Suppose $X_n$ are identically distributed, and their distribution function is $F(x) = 1-x^{-\alpha}$ for $x > 1$ and $F(x) = 0$ for $x \leq 1$, with some $\alpha > 0$. Show that $M_n/n^{1/\alpha}$ converges in distribution, and find the limit.