Homework 2

Probability Theory (MATH 235B, Winter 2008)

1. Consider random variables $X_1, X_2, \ldots$ with distribution functions $F_1, F_2, \ldots$. Suppose that $X_n \to X$ in distribution, and the distribution function F of X is continuous. Prove that $F_n \to F$ in the sup norm, i.e.

$$\|F_n - F\|_\infty := \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \to 0.$$

2. Consider integer valued random variables $X_1, X_2, \ldots$, and a random variable X. Show that $X_n \to X$ in distribution if and only if

$$\mathbb{P}(X_n = k) \to \mathbb{P}(X = k) \qquad \text{for all } k \in \mathbb{Z}.$$

3. **(Uniqueness Lemma)** Suppose that random variables X_n converge to some random variable X in distribution. Show that the distribution of X is uniquely defined. (Hint: you may use Skorokhod's Represenation Theorem.)

4. Suppose $X_n \to X$ in distribution, and a_n are real numbers such that $a_n \to 0$ as $n \to \infty$. Show that $a_n X_n \to 0$ in distribution.

5. **(Convergence of maxima)** Let $X_1, X_2, \ldots$ be independent random variables, and let

$$M_n := \max_{k \le n} X_k, \qquad n = 1, 2, \ldots$$

Suppose X_n are identically distributed, and their distribution function is $F(x) = 1 - x^{-\alpha}$ for $x > 1$ and $F(x) = 0$ for $x \le 1$, with some $\alpha > 0$. Show that $M_n / n^{1/\alpha}$ converges in distibution, and find the limit.