# Homework 3 Solutions.

## (1)

(a) We will show more: that for every $p>0$,

$$\sup_n \mathbb{E}|X_n|^p < \infty \quad \text{implies tightness of } (X_n).$$

Indeed, assume that $\mathbb{E}|X_n| \le C < \infty$ for all $n$.

Then, by Chebychev's inequality,

$$\mathbb{P}(|X_n|^p > t) \le \frac{C}{t} \qquad \text{for all } t>0, \quad n=1,2,\dots$$

Given $\varepsilon>0$, choose $t>0$ s.t. $\frac{C}{t} = \varepsilon$ (hence $t = \frac{C}{\varepsilon}$),

thus

$$\mathbb{P}\left(|X_n| > \left(\frac{C}{\varepsilon}\right)^{1/p}\right) \le \varepsilon \qquad \text{for all } n.$$

This implies tightness of $X_n$.

(b). Choose $(X_n)$ such that $\mathbb{E}|X_n| \to \infty$ but $\sup_n \mathbb{E}|X_n|^{1/2} < \infty$.

(For example, let $X_n$ be s.t. $\mathbb{P}(X_n = n^2) = \frac{1}{n}$, $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n}$).

Then by (a), the sequence $(X_n)$ is tight. QED.

## (4)

By Problem 5 of HW 1, $X_n \to c$ in probability if and only if $X_n \to c$ in distribution. Then the statement follows from the Continuity Theorem for characteristic functions.

②

(a) Let $z \in \mathbb{R}$.

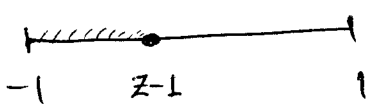
$$\mathbb{P}(X+Y \le z) = \mathbb{P}(X+Y \le z \mid X=1)\cdot \mathbb{P}(X=1) + \mathbb{P}(X+Y\le z \mid X=-1)\mathbb{P}(X=-1)$$

$$= \mathbb{P}(Y \le z-1 \mid X=1)\cdot \tfrac{1}{2} + \mathbb{P}(Y \le z+1 \mid X=-1)\cdot \tfrac{1}{2}$$

$$= \tfrac{1}{2}\mathbb{P}(Y \le z-1) + \tfrac{1}{2}\mathbb{P}(Y \le z+1).$$

If $z \in [0,2)$ then $\mathbb{P}(Y \le z+1) = 0$, while

$$\mathbb{P}(Y \le z-1) = \frac{(z-1)-(-1)}{2} = \frac{z}{2}+1.$$

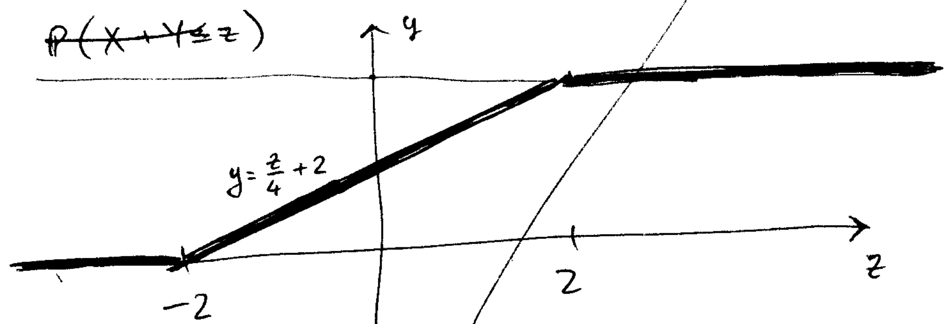



Hence for $z \in [0,2]$,

$$\mathbb{P}(X+Y \le z) = \tfrac{1}{2}\left(\tfrac{z}{2}+1\right) = \tfrac{z}{4}+2. \qquad (*)$$

Similarly, $(*)$ holds for $z \in [-2,0]$, ~~and for z outside [-2,2]~~,

~~$\mathbb{P}(X+Y\le z)$~~




Q.E.D.

This is the distr. function of a uniform distribution on $[-2,2]$

(b). By independence, $\varphi_{X+Y}(t) = \varphi_X(t)\cdot\varphi_Y(t)$. $(**)$

By (a) and using the formula for the ch.f. of a uniform distribution, (and Bernoulli)

we rewrite $(**)$ as

$$\frac{\sin(2t)}{2t} = \cos t \cdot \frac{\sin t}{t}$$

$$\Leftrightarrow \quad \boxed{\sin(2t) = 2\sin t\cos t.}$$

QED.

③

"Only if" part: Assume $X$ takes values $an+b$, $n \in \mathbb{Z}$.

Then $n$ is a random variable (integer valued).

$$\varphi(t) = \mathbb{E}e^{it(an+b)} = e^{itb} \cdot \mathbb{E}e^{itan}$$

Choose $t = \frac{2\pi}{a}$. Then $\mathbb{E}e^{itan} = \mathbb{E}e^{2\pi i n} = \mathbb{E}1 = 1$,

while $|e^{itb}| = 1$ Hence $|\varphi(t)| = 1$.

"If" part. Assume $|\mathbb{E}e^{itX}| = 1$ for some $t \neq 0$

Then $\mathbb{E}e^{itX} = e^{itb}$ for some $b \in \mathbb{R}$.

Equivalently,
$$\mathbb{E}e^{it(X-b)} = 1.$$

By Euler's formula,
$$\mathbb{E}\cos t(X-b) = 1, \qquad \mathbb{E}\sin t(X-b) = 0.$$

Therefore, using the first identity, we have
$$\mathbb{E}\left(1-\cos t(X-b)\right) = 0.$$

The integrand is non-negative ~~for all~~ pointwise, hence
$$1-\cos t(X-b) = 0 \quad \text{a.s.}$$

Hence $t(X-b) = 2\pi n$, $n \in \mathbb{Z}$.

Equivalently, $X = \left(\frac{2\pi}{t}\right)n + b$, $n \in \mathbb{Z}$.

Q.E.D.

④ - see page 1.