Homework 3

Probability Theory (MATH 235B, Winter 2008)

1. Consider a sequence of random variables (X_n).

(a) Show that $\sup_n \mathbb{E}|X_n| < \infty$ implies tightness of the sequence (X_n).

(b) Show that the converse in (a) does not always hold, even if all random variables have finite means.

2. Consider a random variable X such that $\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = \frac{1}{2}$, a random variable Y uniformly distributed in the interval $[-1, 1]$, and suppose that X and Y are independent.

(a) Show that $X + Y$ is uniformly distributed in the interval $[-2, 2]$.

(b) Translate this result into a statement about characteristic functions. You should obtain some well known trigonometric identity.

3. A random variable X has a *lattice distribution* if there exist a real number a and a positive number b such that all values of X have form $a + nb$, $n \in \mathbb{Z}$.

Consider a random variable X with characteristic function φ. Prove that X has a lattice distribution if and only if $|\varphi(t)| = 1$ for some $t \neq 0$.

4. Let X_n be random variables with distribution functions F_n and characteristic functions φ_n. Let c be a constant. Prove that $X_n \to c$ in probability if and only if $\varphi_n(t) \to e^{itc}$ for every $t \in \mathbb{R}$.