Homework 4 Solutions

1

Corollary 21.1 in the Course notes states that, for a r.v. X,

$$\varphi_{-X}(t) = \overline{\varphi_X(t)} \qquad (*)$$

Therefore, if X is symmetric then $\varphi_X(t) = \overline{\varphi_X(t)}$, hence $\varphi_X(t) \in \mathbb{R}$.

Conversely, if $\varphi_X(t) \in \mathbb{R}$ then $(*)$ yields

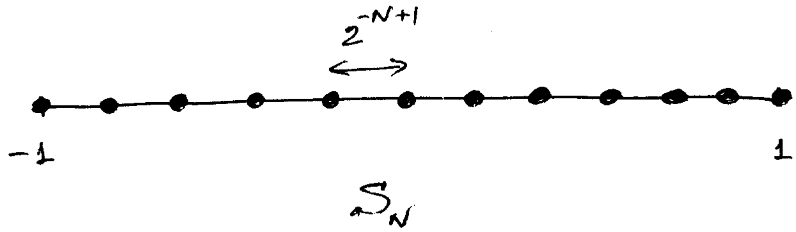
$$\varphi_{-X}(t) = \varphi_X(t)$$

so, by the Inversion Formula, X and $-X$ have the same distribution; thus X is symmetric.

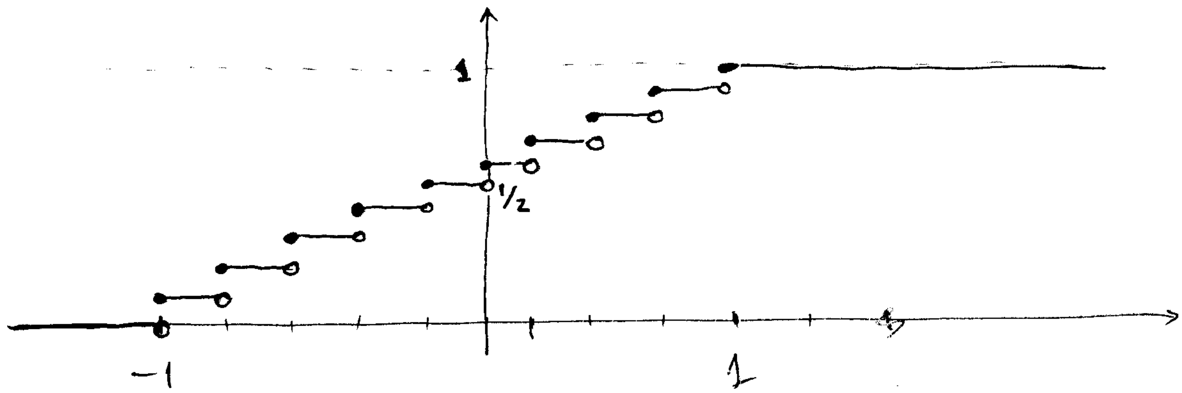
2

(a) Let $S_N := \sum_{n=1}^{N} \frac{X_n}{2^n}$.

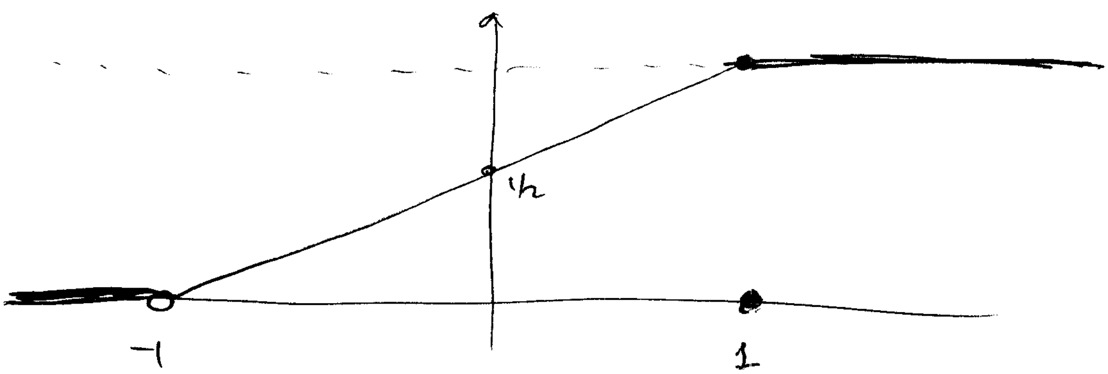
Note that S_N takes values $\{-1, -1+2^{-N+1}, -1+2\cdot 2^{-N+1}, \dots, 1-2^{-N+1}, 1\}$, i.e. ~~the va~~ has 2^N values evenly spaced on the interval $[-1, 1]$. Each value corresponds uniquely to a realization of the random variables $(X_1, \dots X_n)$. Hence each such value is attained with ~~pro~~ (equal) probability 2^{-N}.



Therefore, the distribution function of S_N has this form:



This function obviously converges pointwise to



which is the distribution function of the uniform distribution on $[-1, 1]$. Hence $S_n \to$ uniform distribution on $[-1,1]$.

QED.

(b) Follows by taking the characteristic functions of S_N and X_n, using the multiplicativity property and the Continuity Thm.

(c) If Y_n is Bernoulli $(\frac{1}{2})$, then $\sum_{n=1}^{N} \frac{Y_n}{2^n} = 0.Y_1 Y_2 Y_3 \ldots Y_N$; the right hand side denotes a binary representation.
By (a) and rescaling, these numbers (with Y_k chosen at random) are good approximations of a random number uniformly chosen in $[0,1]$.

(3)

The ch. f. of a Cauchy r.v. X is

$$\varphi_X(t) = e^{-|t|}.$$

Hence $\varphi_{X/n}(t) = \varphi_X(t/n) = e^{-\frac{|t|}{n}}$.

Therefore we have for $S_n = X_1 + \dots + X_n$:

$$\varphi_{S_n/n}(t) = \prod_{k=1}^{n} \varphi_{X_k/n}(t) = \prod_{k=1}^{n} e^{-|t|/n} = e^{-|t|},$$

which is the ch.f. of a Cauchy distribution.

By the Inversion Formula, the proof is complete.

(4)

(a) It suffices to show that, for a r.v. X, we have

$$\left|\varphi(t+h) - \varphi(t)\right| \le 2\,\mathbb{P}(|X| \geq M) + |h| M \qquad (*).$$

for arbitrary $h \in \mathbb{R}$, $M > 0$.

(Then use this inequality for X_n, with M chosen s.t. $\mathbb{P}(|X_n| > M) < \varepsilon$ $\forall n$ by tightness).

Now we prove $(*)$.

$$\begin{aligned} |\varphi(t+h) - \varphi(t)| &\le \mathbb{E}\left|e^{i(t+h)X} - e^{itX}\right| && \text{by Jensen's ineq.} \\ &= \mathbb{E}\left|e^{itX}(e^{ihX} - 1)\right| \\ &= \mathbb{E}\left|e^{ihX} - 1\right|. && \text{(note that } t \text{ disappeared!)} \end{aligned}$$

Now we break the expectation into two parts, for small X and for large X:

1) $$\mathbb{E}\left|e^{ihX}-1\right|\cdot \mathbb{1}_{\{|X|\le M\}} \le \mathbb{E}\left|hX\right|\cdot \mathbb{1}_{\{|X|\le M\}} \le |h|\cdot M.$$

(use $|e^{ix}-1| \le |x|$)

2) $$\mathbb{E}\left|e^{ihX}-1\right|\cdot \mathbb{1}_{\{|X|>M\}} \le \mathbb{E}\, 2\cdot \mathbb{1}_{\{|X|>M\}} = 2\cdot \mathbb{P}\{|X|>M\}.$$

use $|e^{ix}-1| \le |e^{ix}|+1=2$

This completes the proof.

(b) Since (X_n) converges in distribution, it is tight, and $\varphi_n(t)\to\varphi(t)\ \forall t$ (By Continuity Thm).
By (a), (φ_n) are uniformly equicontinuous.
These two properties imply uniform convergence.