1. Corollary 21.1 in the course notes states that, for a r.v. \( X \),
\[ \Psi_X(t) = \overline{\Psi_X(t)} \quad \text{(*)} \]
Therefore, if \( X \) is symmetric then \( \Psi_X(t) = \overline{\Psi_X(t)} \), hence \( \Psi_X(t) \in \mathbb{R} \).
Conversely, if \( \Psi_X(t) \in \mathbb{R} \) then (*) yields
\[ \Psi_{-X}(t) = \Psi_X(t) \]
so, by the Inversion Formula, \( X \) and \( -X \) have the same distribution; thus \( X \) is symmetric.

2. Let \( S_N = \frac{\sum_{n=1}^{N} X_n}{2^n} \).
Note that \( S_N \) takes values \{\(-1, -1+2^{-N+1}, -1+2.2^{-N+1}, \ldots, 1-2^{-N+1}, 1\)\}, i.e. \( S_N \) has \( 2^N \) values evenly spaced in the interval \([-1, 1]\).
Each value corresponds uniquely to a realization of the random variables \((X_1, \ldots, X_N)\). Hence each such value is attained with \( \text{equal probability} \ 2^{-N} \).

\[
\begin{array}{c}
\begin{array}{c}
\vdots \\
-1 \\
S_N \\
1 \\
\vdots \\
\end{array} \\
\begin{array}{c}
\vdots \\
2^{-N+1} \\
\leftrightarrow \\
\end{array} \\
\end{array}
\]
Therefore, the distribution function of $S_n$ has this form:

![Graph of the distribution function]

This function obviously converges pointwise to

![Graph showing convergence]

which is the distribution function of the uniform distribution on $\mathbb{R}$ $[-1, 1]$. Hence $S_n \rightarrow$ uniform distribution on $[-1, 1)$.

8) Follows by taking the characteristic functions of $S_n$ and $X_n$, using the multiplicativity by property and the Continuity Theorem.

9) If $Y_n$ is Bernoulli ($\frac{1}{2}$), then $\sum_{n=1}^{N} \frac{Y_n}{2^n} = \text{b}$. By (a) and rescaling, these numbers (with $Y_n$ chosen at random) are good approximations of a random number uniformly chosen in $[0,1]$. 

-2-
The c.h.f. of a Cauchy r.v. $X$ is

$$
\varphi_X(t) = e^{-|t|}.
$$

Hence

$$
\varphi_{\frac{X}{n}}(t) = \frac{\varphi_X(t/n)}{n} = e^{-\frac{|t|}{n}}.
$$

Therefore we have for $S_n = X_1 + \ldots + X_n$:

$$
\varphi_{\frac{S_n}{n}}(t) = \prod_{k=1}^n \varphi_{\frac{X_k}{n}}(t) = \prod_{k=1}^n e^{-\frac{|t|}{n}} = e^{-|t|},
$$

which is the c.h.f. of a Cauchy distribution.

By the inversion formula, the proof is complete.

---

It suffices to show that, for a r.v. $X$, we have

$$
|\varphi(t+h) - \varphi(t)| \leq 2P(\{X > M\}) + 1hM \quad (\ast)
$$

for arbitrary $h \in \mathbb{R}$, $M > 0$.
(Then use this inequality for $X_n$, with $M$ chosen s.t. $P(\{X > M\}) < \varepsilon$ by tightness).

Now we prove $(\ast)$.

$$
|\varphi(t+h) - \varphi(t)| \leq E|e^{i(t+h)X} - e^{itX}| \quad \text{by Jensen's inq.}
$$

$$
= E|e^{itX}(e^{ihX} - 1)|
$$

$$
= E|e^{ihX} - 1|.
$$

(note that $t$ disappeared!)
Now we break the expectation into two parts. For small $x$ and for large $x$:

1) $\mathbb{E} |e^{ix} - 1| \cdot 1_{\{|x| \leq M\}} \leq \mathbb{E} |ix| \cdot 1_{\{|x| \leq M\}} \leq |h| \cdot M$.
   \[\text{(use } |e^{ix} - 1| \leq |x|)\]

2) $\mathbb{E} |e^{ix} - 1| \cdot 1_{\{|x| > M\}} \leq \mathbb{E} 2 \cdot 1_{\{|x| > M\}} = 2 \cdot P \{|x| > M\}$.
   \[\text{(use } |e^{ix} - 1| \leq |e^{ix}| + 1 = 2)\]

This completes the proof.

By Continuity Thm:

8) Since $(X_n)$ converges in distribution, it is tight, and $\rho_n(t) \to \rho(t)$.

by (a), $(\rho_n)$ are uniformly equicontinuous.

These two properties imply uniform convergence.