Homework 4

Probability Theory (MATH 235B, Winter 2008)

1. Prove that a random variable X is symmetric if and only if its characteristic function $\varphi(x)$ is real.

2. (a) Let X_n be independent random variables with distribution $\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = 1/2$. Prove that the random variable

$$X = \sum_{n=1}^{\infty} \frac{X_n}{2^n}$$

is uniformly distributed in $[-1, 1]$.

(b) Use characteristic functions to deduce the trigonometric identity

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \cos\left(\frac{x}{2^n}\right).$$

(c) Describe an algorithm that generates a random variable (approximately) uniformly distributed in $[0, 1]$. (Think of choosing digits in a binary representation at random).

3. Prove that if $X_1, \ldots, X_n$ are independent random variables with Cauchy distribution then $(X_1 + \cdots X_n)/n$ is a random variable with Cauchy distribution. (Use characteristic functions).

4. (a) Show that if (X_n) is a tight sequence of random variables, then the characteristic functions (φ_n) are uniformly equicontinuous (for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|s - t| < \delta$ implies $|\varphi_n(s) - \varphi_n(t)| < \varepsilon$ for all n.)

(b) Prove that if a sequence of random variables X_n converges in distribution to X, then $\varphi_{X_n}(t) \to \varphi_X(t)$ uniformly on bounded sets. (You can use a theorem from analysis that equicontinuity and pointwise convergence on a bounded set implies uniform convergence.)